Differential Equations Tutor, Volume I

Worksheet 7

Existence and Uniqueness Theorem

Worksheet for Differential Equations Tutor, Volume I, Section 7:

Existence and Uniqueness Theorem

For the following differential equations:

(a) Does a solution exist?

(b) If a solution exists, is the solution unique? Over what interval (if any) is it unique?

Show all your work. The main point of these exercises is to see the existence and uniqueness theorem in action. To demonstrate that a solution exists, then, you should try to actually find a solution. To demonstrate that a solution $s(t)$ is unique, a good technique is to prove that $A(t)s(t)$ or $A(t) + s(t)$ cannot be a solution for any function $A(t)$. To demonstrate that a solution is not unique, find another solution.

1. $\frac{dx}{dt} = 2x$ with $x(0) = 3$;

2. $\frac{dy}{dt} + ty = t$ with $y(0) = 100$;

3. $\frac{dx}{dt} = 2x + 1$ with $x'(0) = 4$. Hint: how can you demonstrate that $x'(0) = 4$ is equivalent to an initial condition of type $x(t_0) = x_0$?

4. $\frac{dx}{dt} = \frac{2t}{e^x}$ with $x(0) = 0$. Hint: use a transformation (equivalent to a u-substitution) to put this equation into the form $\frac{du}{dt} = 2t$, so that you can then apply the existence and uniqueness theorem. If a solution for u exists on some interval , what does that imply about x?

5. $2xdx + ydy = 0$ with $y(1) = 3$;

6. $\frac{dx}{dt} = \frac{x}{t}$ with $x(-1) = -2$;

7. $\frac{dy}{dx} = y \ln x$ with $y(100) = 5$;

8. $\frac{dx}{dt}+\frac{x}{t^2-1}=\frac{2}{2t-1}$ with $x(0)=0$. What if instead of $x(0)=0$ we have $x(2)=0$?

9. $\frac{dx}{dt}+x\tan t=1$ with $x(3)=5$. What if instead of $x(3)=5$ we have $x(5)=5000$?

10. $\frac{dy}{dx}+y\csc\frac{x}{2}=x$ with $y(-3)=4$. What if instead of $y(-3)=4$ we have $y(3)=4$?

Answer key.

1. $\frac{dx}{dt} = 2x$ with $x(0) = 3$.

This is a fairly basic differential equation. The existence-uniqueness theorem says first that a solution exists, and second that there is only one continuous function that solves this differential equation with initial condition. There is only one continuous function that solves this differential equation with initial condition over any domain. The behavior of any solution function is prescribed for the entire real line just based on the initial condition. We can put the differential equation in form

$$\frac{dx}{dt} + p(t)x = q(t)$$

which in this case is

$$\frac{dx}{dt} - 2x = 0$$

Since $p(t) = -2$ and $q(t) = 0$ are continuous over the entire real line, then there is one continuous function that solves this differential equation over the entire real line. This does not mean that there are other continuous functions that solve this differential equation with initial condition over smaller spaces. No other solution is defined anywhere on the real line.

This sounds like an abstract discussion, but it becomes clear as we begin to work the problem. Even without the statement of the existence and uniqueness theorem, we know how to find a solution (proving that it exists), and we can also prove that the solution is unique. We can corroborate the formal statement of the existence and uniqueness theorem by demonstrating, in this case, that all the assertions made by that theorem are true.

First we want to show that a solution to this differential equation exists. The most straightforward way to show that a solution exists is to actually find the solution. We can easily find the solution because this differential equation is separable. Since

$$\frac{dx}{dt} = 2x$$

we have that

$$\frac{1}{x}dx = 2dt$$

Integrating both sides,

$$\int \frac{1}{x}dx = \int 2dt$$

so

$$\ln|x| = 2t + C$$

where we move the constant to the right-hand side. Exponentiating, then,

$$|x| = e^{2t+C} = Ce^{2t}$$

We can drop the absolute value, and in this section it is important to understand why. Could there be a solution where at some point $x = Ce^{2t}$ and at another point $-x = Ce^{-2t}$ so $x = -Ce^{-2t}$? It actually turns out that this is impossible if $x(t)$ is to be continuous. If $x = Ce^{2t}$ at some point and $-Ce^{2t}$ at another point, it follows that there is some point a where on one side of a, $x = Ce^{2t}$ and on another side of a, $x = -Ce^{2t}$ - more formally,

$$\lim_{t\to a^+} x(t) = Ce^{2t}, \lim_{t\to a^-} x(t) = -Ce^{2t}$$

or vice-versa. However, then $x(t)$ is discontinuous at a with a "jump discontinuity." There is a discontinuity since

$$Ce^{2t} \neq -Ce^{-2t}$$

We know that these two can never be equal since

$$e^{2t} \neq 0$$

for any value of t, unless $C = 0$ which is the constant "steady-state" solution. Then, since x cannot equal both Ce^{2t} and $-Ce^{2t}$, we can simply define

$$x = Ce^{2t}$$

whether C is positive or negative. Alternatively, we can reason that

$$x(t) = Ce^{2t} \neq 0$$

so it is either positive or negative. Then, we sign C accordingly and we can drop the absolute value to get

$$x(t) = Ce^{2t}$$

Now we can apply the initial condition:

$$x(0) = 3$$

so

$$Ce^{2(0)} = 3$$

and

$$C = 3$$

leaving us with the specific solution

$$x(t) = 3e^{2t}$$

This specific solution is defined everywhere and solves the differential equation everywhere (not just close to $t = 0$), which corroborates the result of the existence theorem because $p(t) = -2$ and $q(t) = 0$ are continuous everywhere.

The next step is to prove that this solution is unique - that there is no other function $X(t)$ that solves the differential equation with initial condition anywhere in the neighborhood of $t = 0$ - let alone over the entire real line. We suppose that such a function $X(t)$ existed with $X(0) = 3$. Then, we must have

$$X(t) = a(t)x(t)$$

for some function $a(t)$, because

$$\frac{X(t)}{x(t)} = a(t)$$

for some function $a(t)$. The function $a(t)$ is well-defined since $x(t) = 3e^{2t}$ is never zero, so we can comfortably divide by it. Then, we test to see under what conditions for a the function

$$X(t) = 3e^{2t}a(t)$$

could solve our original differential equation

$$\frac{dx}{dt} = 2x$$

We have

$$\frac{d}{dt}X(t) = \frac{d}{dt}\left(3e^{2t}a(t)\right) = 6e^{2t}a(t) + 3e^{2t}a'(t)$$

so we want to know under what conditions

$$6e^{2t}a(t) + 3e^{2t}a'(t) = 6e^{2t}a(t)$$

This is true just when

$$3e^{2t}a'(t) = 0$$

Since e^{2t} is never zero, this is true when

$$a'(t) = 0$$

which means that $a(t)$ must be a constant. Then, the only other possible solutions are

$$X(t) = a \cdot 3e^{2t}$$

This matches the notion that the general solution to this differential equation is Ce^{2t}. However, the presence of the initial condition "winnows down" the general solution, which we suspect will disallow any multiplicative constant. In fact, for

$$X(0) = 3$$

we would need

$$a \cdot 3e^0 = 3$$

which means

$$a = 1$$

Then, we have proven that the *only* specific solution is

$$x(t) = 3e^{2t}$$

The answer, then, given by the existence and uniqueness theorem and corroborated by the differential equations techniques that we already know, is that

there exists a unique solution to this differential equation with initial condition.

Furthermore,

This is a solution, and the unique solution, over the entire real line.

The theorem is applicable to differential equations that are more difficult to solve, or even those where we can't actually find the solution. The theorem does not depend on a constructive proof such as we were able to employ here.

2. $\frac{dy}{dt} + ty = t$ with $y(0) = 100$.

This differential equation is perfectly suited to use the existence and uniqueness theorem. The equation is in the form

$$\frac{dy}{dt} + p(t)y = q(t)$$

with

$$p(t) = q(t) = t$$

We have that $p(t)$ and $q(t)$ are continuous and defined over the entire real line. There also is an initial condition provided. Then, by the existence and uniqueness theorem,

there exists a unique solution to this differential equation over the entire real line.

There are no other solutions that meet the initial condition that are defined anywhere, that differ from the unique solution over any part of the real line. We can assert this with confidence even though the initial condition is "large" (a large value of x), and even though we have not actually found a solution or demonstrated its uniqueness.

Having asserted the existence and uniqueness theorem, we can find the unique solution. This is a linear nonhomogeneous equation. We first need to solve the related homogeneous equation,

$$\frac{dy}{dt} + ty = 0$$

This is a separable equation:

$$\frac{1}{y}dy = -tdt$$

so

$$\ln|y| = -\frac{t^2}{2} + C$$

and

$$|y| = Ce^{-\frac{t^2}{2}}$$

Once again, the solution to the homogeneous solution is not continuous as it must be if C changes in value. Then, C must be either positive or negative, so we can drop the absolute value and find that

$$y = ke^{-\frac{t^2}{2}}$$

This is the related homogeneous solution. We posit a solution to the nonhomogeneous equation of the form

$$y(t) = k(t)e^{-\frac{t^2}{2}}$$

Then,

$$\frac{dy}{dt} = k'(t)e^{-\frac{t^2}{2}} - tk(t)e^{-\frac{t^2}{2}}$$

We solve

$$\frac{dy}{dt} + ty = t$$

This is

$$k'(t)e^{-\frac{t^2}{2}} = t$$

so

$$k'(t) = te^{\frac{t^2}{2}}$$

and

$$k(t) = \int te^{\frac{t^2}{2}}dt = e^{\frac{t^2}{2}} + C$$

using a u-substitution of

$$u = \frac{t^2}{2}$$

Then we have

$$y(t) = k(t)h(t) = \left(e^{\frac{t^2}{2}} + C\right)\left(e^{-\frac{t^2}{2}}\right) = 1 + Ce^{-\frac{t^2}{2}}$$

Since

$$y(0) = 100$$

we have

$$100 = 1 + Ce^0$$

so

$$C = 99$$

and we have constructed a specific solution over the real line (demonstrating existence):

$$y(t) = 1 + 99e^{-\frac{t^2}{2}}$$

Suppose another specific solution existed, $X(t)$. Then we would have

$$X(t) = 1 + 99e^{-\frac{t^2}{2}} + a(t)$$

for some function $a(t)$ with

$$a(0) = 0$$

If $X(t)$ satisfies the differential equation, we have

$$\frac{dX}{dt} + tX = t$$

so

$$1 - 99te^{-\frac{t^2}{2}} + a'(t) + t + 99te^{-\frac{t^2}{2}} + ta(t) = t$$

which means

$$a'(t) + ta(t) = 0$$

This is separable, so

$$\frac{1}{a}da = -t, \ln|a| = -\frac{t^2}{2} + C, a = Ce^{-\frac{t^2}{2}}$$

Since $a(0) = 0$ we must have

$$C = 0$$

proving the uniqueness of the solution that we have found. It is unique over the entire real line since no other specific solution is defined anywhere.

3. $\frac{dx}{dt} = 2x + 1$ with $x'(0) = 4$;

We can adapt the argument of the existence-uniqueness theorem to support this problem. We put the equation into the required form

$$\frac{dx}{dt} + p(t)x = q(t)$$

Here, the equation is

$$\frac{dx}{dt} - 2x = 1$$

so

$$p(t) = -2, q(t) = 1$$

Both of these functions are continuous everywhere, so that means that there exists a solution that is continuous everywhere. How about uniqueness? We are not, exactly, given an initial condition since the condition we are given is on x', not on x. We could directly apply the uniqueness theorem if $x(0) = 4$, for instance. Since $x'(0) = 4$ that does indicate that there is a specific solution, so there is an initial value at some point. We can then argue from the existence-uniqueness theorem that the solution is unique.

More formally, we can prove, "by construction," that the solution exists - and then find the initial condition in the form $x(t_0) = x_0$ that allows us to apply the existence-uniqueness theorem formally and directly. We need to solve the equation

$$\frac{dx}{dt} = 2x + 1$$

The standard form of this equation is

$$\frac{dx}{dt} - 2x = 1$$

This is a nonhomogeneous linear ODE. To solve a nonhomogeneous equation, we first need to solve the related homogeneous equation,

$$\frac{dx}{dt} - 2x = 0$$

As we solved above, the general solution to the related homogeneous equation is

$$x(t) = ke^{2t}$$

Then, varying the parameter, we look for a solution to the nonhomogeneous equation in the form

$$k(t)e^{2t}$$

We have

$$\frac{dx}{dt} - 2x = k'(t)e^{2t} + 2k(t)e^{2t} - 2k(t)e^{2t}$$

which we set equal to 1:

$$k'(t)e^{2t} = 1, k'(t) = e^{-2t}$$

so

$$k(t) = -\frac{1}{2}e^{-2t} + C$$

for a constant of integration C. Then,

$$x(t) = \left(-\frac{1}{2}e^{-2t} + C\right)e^{2t} = -\frac{1}{2} + Ce^{2t}$$

Now we apply

$$x'(0) = 4$$

We have

$$x'(t) = 0 + 2Ce^{2t}$$

Then,

$$x'(0) = 0 + 2Ce^{2(0)} = 4$$

so

$$C = 2$$

and the specific solution is

$$-\frac{1}{2} + 2e^{2t}$$

We now have an initial value in the form $x(t_0) = x_0$; say,

$$x(0) = \frac{3}{2}$$

so by the existence-uniqueness theorem there is a unique solution. Suppose that there were another solution

$$X(t) = x(t) + a(t)$$

Then, we would have

$$\frac{dX}{dt} = \frac{dx}{dt} + \frac{da}{dt} = 4e^{2t} + \frac{da}{dt}$$

and

$$2X + 1 = 2\left(-\frac{1}{2} + 2e^{2t} + a(t)\right) + 1 = 4e^{2t} + 2a(t)$$

Then, for the differential equation to hold true,

$$\frac{da}{dt} = 2a(t)$$

which is solved by

$$a(t) = Ce^{2t}$$

for some constant C. This matches the fact that the general solution to the differential equation is

$$x(t) = -\frac{1}{2} + Ce^{2t}$$

However, $x'(0) = 4$ is met only when $C = 4$. Then, we have demonstrated uniqueness to corroborate the existence-uniqueness theorem. As before, the domain of uniqueness or interval of validity is the whole real line.

$\boxed{\textbf{There exists a unique solution over the entire real line.}}$

4. $\frac{dx}{dt} = \frac{2t}{e^x}$ with $x(0) = 0$.

The existence and uniqueness theorem indicates whether or not a solution exists and whether or not it is unique. To apply this theorem, however, strictly speaking, we need a differential equation in the form

$$\frac{dx}{dt} + p(t)x = q(t)$$

This equation cannot be manipulated into that form by algebraic operations, so it seems that we may be at a loss to apply the formal existence theorem. If we want to apply the theorem formally, we need to use an *transform.* We let

$$u = e^x$$

Then,

$$\frac{du}{dt} = \frac{d}{dt}e^x = e^x\frac{dx}{dt}$$

so

$$\frac{dx}{dt} = e^{-x}\frac{du}{dt} = \frac{1}{u}\frac{du}{dt}$$

and the differential equation in terms of u and t is

$$\frac{1}{u}\frac{du}{dt} = \frac{2t}{u}$$

Then,

$$\frac{du}{dt} = 2t$$

which is a differential equation in u and t subject to the existence and uniqueness theorem. There exists a unique solution function subject to the initial condition (which is $u(0) = e^0 = 1$) and the interval of validity is likewise the entire real line since $p(t) = -2t$ is continuous everywhere. There is a problem, however: u can only be positive since $u = e^x$. So the unique solution is only defined for positive u- and then a unique solution $x = \ln u$ is defined. But it so happens that the general solution for u is

$$u = t^2 + C$$

and

$$C = 1$$

so

$$u = t^2 + 1$$

is positive for any value of t. Then the interval of validity for existence and uniqueness is the entire real line. In this convoluted way, we can apply the existence and uniqueness theorem to this problem even if it is not yet in the desired form.

We can also prove existence and uniqueness using the basic differential equations techniques that we have already developed. Even though this differential equation is not linear, it is still separable. We write it as

$$e^x dx = 2t dt$$

Integrating, then

$$e^x = t^2 + C$$

so

$$x(t) = \ln\left(t^2 + C\right)$$

This is the general solution to this differential equation. We have

$$x(0) = 0$$

so

$$x(0) = \ln\left(0^2 + C\right) = 0$$

so

$$\ln C = 0$$

which means

$$C = 1$$

and the specific solution is

$$x(t) = \ln\left(t^2 + 1\right)$$

Then, there is a solution to the differential equation that exists over the entire real line, since $t^2 + 1$ is positive everywhere and its logarithm is defined.

We can also prove that the solution $x(t) = \ln(t^2 + 1)$ is unique. We can prove that there is no other continuous solution to this differential equation anywhere in the vicinity of $t = 0$, or over the entire real line. Suppose that such another solution $X(t)$ solved the differential equation. We can write

$$X(t) = \ln\left(t^2 + a(t)\right)$$

for some $a(t)$. Then,

$$a(t) = e^{X(t)} - t^2$$

which is well-defined everywhere for all values of t, X. We have constructed the form of any possible other solution $X(t)$ by a unique construction, writing it in terms of $a(t)$. Then, we have

$$\frac{dX}{dt} = \frac{1}{t^2 + a(t)} \cdot (2t + a'(t))$$

and we have

$$\frac{2t}{e^X} = \frac{2t}{t^2 + a(t)}$$

For the differential equation to hold true, we need

$$\frac{2t + a'(t)}{t^2 + a(t)} = \frac{2t}{t^2 + a(t)}$$

so

$$\frac{a'(t)}{t^2 + a(t)} = 0$$

If a is a continuous function, then,

$$a'(t) = 0$$

and a is a constant, so every solution to the differential equation is of the form

$$x(t) = \ln\left(t^2 + a\right)$$

This corresponds to the form of the general solution. To meet the initial condition, however, we must have $a = 1$. Then there is no other solution to the differential equation with initial condition defined anywhere. It is not possible, for instance, to have a solution in the form of $x(t) = \ln(t^2 + 1)$ near the origin and $x(t) = \ln(t^2 + a)$ somewhere else. If this were the case, there would be a place of discontinuity where $\ln(t^2 + 1)$ meets $\ln(t^2 + a)$, and these are never equal unless $a = 1$. Then, no continuous function is defined that is different from $x(t) = \ln(t^2 + a)$ anywhere. We have proven, then, that

there exists a unique solution everywhere.

Our calculations corroborate the result of the existence and uniqueness theorem.

5. $2xdx + ydy = 0$ with $y(1) = 3$.

The form of this problem is, we note, different than the others. To apply the existence-uniqueness theorem, we would need to put the equation in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

for some functions $p(x)$ and $q(x)$. If we try to put the function into this form, however, we have that

$$ydy = -2xdx$$

so

$$\frac{dy}{dx} = \frac{-2x}{y}$$

and

$$\frac{dy}{dx} + 2x \cdot \frac{1}{y} = 0$$

Here, it is not possible to put the differential equation in the desired form for application of the theorem, and if we try we encounter a problem with a discontinuity at $y = 0$. Is there any transformation that could make this problem subject to the existence-uniqueness theorem? No basic transformation seems to work, so the existence-uniqueness theorem as we have it actually doesn't indicate whether there exists a solution or whether the solution is unique.

This differential equation, however, is simple enough to be solved, which would demonstrate existence and possibly hint at uniqueness. We could solve it as an exact differential equation or as a separable differential equation. If we separate, we have that

$$ydy = -2xdx$$

Integrating,

$$\frac{y^2}{2} = -x^2 + C$$

for some constant C. Multiplying by 2, we have that

$$y^2 + 2x^2 = C$$

for some constant C. This is the general solution. If $y(1) = 3$, then we have

$$(x, y) = (1, 3)$$

on this curve, so

$$3^2 + 2(1^2) = C$$

and

$$11 = C$$

so the specific solution is

$$2x^2 + y^2 = 11$$

We have proven that a solution exists, but over what domain? We must have

$$y^2 = 11 - 2x^2 \geq 0$$

so

$$11 - 2x^2 \geq 0, x^2 \leq \frac{11}{2}, -\sqrt{\frac{11}{2}} \leq x \leq \sqrt{\frac{11}{2}}$$

This is the range of existence. No solution to this initial value problem exists that is defined outside of this range. How about uniqueness? At first, it might seem that the solution is not unique because we could have

$$y = \pm\sqrt{11 - 2x^2}$$

However, we are guaranteed by the initial condition to have y positive when $x = 1$, so then y must be positive at that point. It follows that y must be positive everywhere on the interval of existence: if it were to be negative and continuous it must be zero somewhere, but its zeroes are only at the edges of the interval of existence at $x = \pm\sqrt{\frac{11}{2}}$. Then, the solution is unique also on this interval. Then,

$$\boxed{\textbf{There exists a unique solution on the interval } -\frac{\sqrt{22}}{2} \leq x \leq \frac{\sqrt{22}}{2}}$$

We could also solve this problem by transformation. We could let

$$u = 11 - 2x^2$$

which means that $u \leq 11$. Then,

$$\frac{du}{dy} = -4x\frac{dx}{dy}$$

so

$$\frac{dy}{du} = -\frac{1}{4x}\frac{dy}{dx}$$

and

$$\frac{dy}{dx} = -4x\frac{dy}{du}$$

so the differential equation

$$\frac{dy}{dx} + 2x \cdot \frac{1}{y} = 0$$

becomes

$$-4x\frac{dy}{du} + 2x \cdot \frac{1}{y} = 0$$

so

$$\frac{dy}{du} = \frac{1}{2y}$$

If we are still hoping to use the existence theorem, we could use

$$w = y^2$$

which means that $w \geq 0$. Then,

$$\frac{dw}{du} = 2y\frac{dy}{du}$$

so

$$\frac{dy}{du} = \frac{1}{2y}\frac{dw}{du}$$

and

$$\frac{1}{2y}\frac{dw}{du} = \frac{1}{2y}$$

so

$$\frac{dw}{du} = 1$$

The initial condition

$$x = 1, y = 3$$

implies

$$u = 9, w = 9$$

Then, there exists a unique solution for v in terms of u over the interval of validity where they both exist, namely

$$u = w$$

We know that

$$w \geq 0$$

which means

$$u \geq 0$$

which translates into the same interval of validity that we found, since $u = 11 - 2x^2$.

6. $\frac{dx}{dt} = \frac{x}{t}$ with $x(-1) = -2$.

This problem is of a different type. It is relatively easy to put this problem into the desired form

$$\frac{dx}{dt} + p(t)x = q(t)$$

We have

$$\frac{dx}{dt} + \left(\frac{-1}{t}\right) x = 0$$

Then, the existence-uniqueness theorem tells us that there exists a unique solution on the largest domain where $p(t)$ and $q(t)$ are continuous that contains the initial condition. The function $q(t) = 0$ is continuous everywhere, but $p(t) = -\frac{1}{t}$ is not continuous at $t = 0$. Then, there exists a unique solution on any domain not containing zero and the initial condition, which is at $t = -1$. There exists a unique solution on the range $-3 \leq t \leq 0$, for instance, or on the range $-5 \leq t < 0$. The largest possible domain that contains the point of initial condition

$$t = -1$$

is

$$-\infty < t < 0$$

The same unique solution that exists and is defined on any of these smaller interval is also defined on the larger interval, so the existence-uniqueness theorem tells us. But the existence-uniqueness theorem says that this solution is not unique outside of this domain, because there is a break in continuity and the solution could be different for $t > 0$. We can state that

$$\boxed{\textbf{There exists a unique solution on } -\infty < t < 0}$$

as a result of the existence and uniqueness theorem, and then proceed to actually solve the differential equation to see exactly what these assertions signify.

This differential equation is separable, so we can solve it: we have

$$\frac{1}{x}dx = \frac{1}{t}dt$$

Integrating, then, we have

$$\ln|x| = (\ln|t|) + C$$

so

$$|x| = e^{(\ln|t|)+C} = C|t|$$

We cannot remove the absolute values here, as we did in previous sections, because the absolute values are key to the restrictions on the interval of validity. What if, for instance, $x = t$ for negative t, and $x = -t$ for positive t? This might create a problem at $t = 0$ because the function would not be smooth - but we recall that the original differential equation is not defined for $t = 0$. Applying the initial condition

$$x(-1) = -2$$

we have

$$|-2| = C\,|-1|$$

so

$$C = 2$$

and we see that

$$|x| = 2|t|$$

We can also process the double absolute value. Actually,

$$x = 2|t|$$

is not a solution to the initial value problem because at $t = -1$ we need $x = -2$ so the only solution to the initial value problem is

$$x = -2|t|$$

Now, when t is negative, the solution takes the form

$$x = -2(-t) = 2t$$

which is the specific solution. This clearly exists and solves the differential equation for all $-\infty < t < 0$, which is an interval of validity around $t = -1$. What about when t is positive? The case when t is positive is the key as to why the interval of validity does not go past zero. Although the specific solution $x = 2t$ might be defined at $t = 0$, it is not a solution to the differential equation at $t = 0$ because the differential equation is not defined at $t = 0$. So the solution does not exist at $t = 0$. At $t > 0$, the solution exists, but it is not unique. We could have

$$x(t) = \begin{cases} 2t & t < 0 \\ -2t & t > 0 \end{cases}$$

or actually $x(t)$ defined for $t > 0$ as Ct for any C. This is another solution to the differential equation $x(t)$ that is defined for all $t \neq 0$. It solves the differential equation everywhere and it is continuous on its domain. There is no discontinuity at zero because the solution is not defined at $t = 0$. Then, there is another solution different than $x(t) = 2t$ that is continuous and defined on a larger range than the interval of uniqueness. The uniqueness result in the theorem only states that, in the interval of the initial condition where $p(t)$ and $q(t)$ are continuous, there will be a unique solution. We have demonstrated this by finding the unique solution, and showing that outside the range of validity the solution is not unique. But inside the range of validity, there is no other solution to the initial value problem.

7. $\frac{dy}{dx} = y \ln x$ with $y(100) = 5$.

The logic of this problem is similar to the last problem, except even more clear-cut. This differential equation easily takes the form

$$\frac{dy}{dx} - y \ln x = 0$$

so

$$p(x) = \ln x, q(x) = 0$$

The function $p(x) = \ln x$ is continuous everywhere it is defined, but it is only defined for

$$0 < x < \infty$$

Then we only expect a solution that is defined for $0 < x < \infty$, but we expect our solution to be unique since there is an initial condition provided:

$$\boxed{\textbf{There is a unique solution defined for } 0 < x < \infty}$$

We can confirm the validity of this result by finding that solution and proving its uniqueness. This equation is separable, so

$$\frac{1}{y} dy = \ln x dx$$

Integrating both sides, then,

$$\ln |y| = \int \ln x dx$$

We solve $\int \ln x dx$ by parts. We note

$$\ln x = (\ln x) \cdot 1$$

so we let

$$U(x) = \ln x, V'(x) = 1$$

and then

$$U'(x) = \frac{1}{x}, V(x) = x$$

and

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 dx = x \ln x - x + C$$

so we have

$$\ln |y| = x \ln x - x + C$$

and

$$|y| = e^{x \ln x - x + C} = Ce^{x \ln x} e^{-x} = Cx^x e^{-x}$$

We can drop the absolute value on y in this case. On the continuous range $x > 0$, it is impossible for $Cx^x e^{-x}$ to be both positive and negative. Unlike the last problem, where there was a break in the range at zero that permitted the constant to have multiple values on different intervals and be continuous on each separate interval, here there is only one interval where a solution can be defined. We never have $y = 0$ since

$$x^x \neq 0, e^{-x} \neq 0$$

so then y is either positive or negative and we can write

$$y(x) = Cx^x e^{-x}$$

Then, we can apply

$$y(100) = 5$$

to get that

$$5 = C \cdot 100^{100} e^{-100}$$

so

$$C = 5 \cdot 100^{-100} e^{100}$$

and the specific solution is

$$y(x) = 5 \cdot 100^{-100} e^{100} x^x e^{-x}$$

This is defined for $x > 0$. If $x < 0$, then x^x is not defined, because a negative number's root is generally not defined over the reals. If $x = 0$, then 0^0 is not defined. But the solution exists for $x > 0$. We could also prove that the solution for $x > 0$ is unique. Suppose that

$$Y(x) = a(x)x^x e^{-x}$$

is a solution to this differential equation. Then, we want to see under what conditions we have

$$\frac{dY}{dx} = Y \ln x$$

We have

$$\frac{dY}{dx} = a'(x)x^x e^{-x} + a(x)\frac{d}{dx}\left(x^x e^{-x}\right)$$

We have

$$x^x = e^{x \ln x}$$

so we have

$$\frac{d}{dx}e^{x\ln x - x} = e^{x\ln x + x}\frac{d}{dx}\left(x \ln x - x\right) = e^{x \ln x + x}\left(\ln x + 1 - 1\right) = \left(\ln x\right) \cdot e^{x \ln x + x}$$

Then,

$$\frac{dY}{dx} = a'(x)x^x e^{-x} + a(x)\ln x e^{x \ln x + x}$$

For this to be equal to

$$Y \ln x = a(x)\left(\ln x\right) e^{x \ln x + x}$$

we need

$$a'(x)x^x e^{-x} = 0$$

which is only true when $a(x)$ is a constant, and only the particular value of the constant that we have found meets the initial condition. So we have corroborated the results for existence and uniqueness.

8. $\frac{dx}{dt} + \frac{x}{t^2-1} = \frac{2}{2t-1}$ with $x(0) = 0$. What if instead of $x(0) = 0$ we have $x(2) = 0$?

This function is already in the form

$$\frac{dx}{dt} + p(t)x = q(t)$$

We have

$$p(t) = \frac{1}{t^2-1}, q(t) = \frac{2}{2t-1}$$

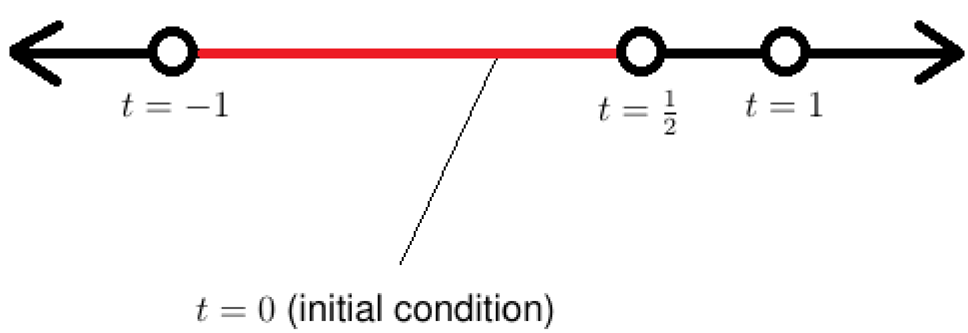
Then, $p(t)$ is continuous except at

$$t = \pm 1$$

and $q(t)$ is continuous except at

$$t = \frac{1}{2}$$

We can then break the real line up into intervals over which this differential equation is defined, where the interval of validity is in red:



Plot for 8

Of the four intervals here defined, the interval that includes the initial condition at $t = 0$ is

$$-1 < t < \frac{1}{2}$$

The differential equation could and does have a solution on the other intervals where it is defined. However, the only unique solution is on this red interval. In other words, there are many solutions to this initial value problem based on all of the piecewise different solutions on other intervals, but every solution is unique on this interval of validity. Then,

$$\boxed{\textbf{For } x(0)=0, \textbf{ there exists a unique solution on } -1<t<\frac{1}{2}}$$

We can demonstrate our results by actually finding the solution. This is a nonhomogeneous linear differential equation. We solve it by first solving the related homogeneous equation:

$$\frac{dx}{dt}+\frac{x}{t^2-1}=0$$

so

$$\frac{dx}{x}=-\frac{dt}{t^2-1}$$

and

$$\ln|x|=\int -\frac{1}{t^2-1}dt$$

This integral is solved by partial fractions. We have

$$t^2-1=(t+1)(t-1)$$

so

$$\frac{1}{t^2-1}=\frac{A}{t+1}+\frac{B}{t-1}$$

for constants A, B. Cross-multiplying by t^2-1, we have

$$1=A(t-1)+B(t+1)$$

so

$$(A+B)t+(-A+B)=1$$

which means

$$A+B=0, -A+B=1$$

This means that

$$B = \frac{1}{2}, A = -\frac{1}{2}$$

and

$$-\frac{1}{t^2 - 1} = -\left(-\frac{\frac{1}{2}}{t+1} + \frac{\frac{1}{2}}{t-1}\right) = \frac{\frac{1}{2}}{t+1} - \frac{\frac{1}{2}}{t-1}$$

and

$$\int -\frac{1}{t^2-1}dt = \int \frac{\frac{1}{2}}{t+1}dt - \int \frac{\frac{1}{2}}{t-1}dt = \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|t-1| + C$$

so we have

$$\ln|x| = \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|t-1| + C = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + C$$

and

$$|x| = Ce^{\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|} = Ce^{\ln\sqrt{\frac{t+1}{t-1}}} = C\sqrt{\left|\frac{t+1}{t-1}\right|}$$

Around $t = 0$, the function $\frac{t+1}{t-1}$ is always negative, so - in this interval - we have

$$|x| = C\sqrt{\frac{t+1}{1-t}}$$

We also have that x can never be zero in this range. We suspect possibly other answers that involve changing signs where y can be zero - that is, around $t = -1$ or on the other side of $t = 1$, but since the signs don't change in the interval of validity and we cannot switch from one branch to another, we have

$$x(t) = k\sqrt{\frac{t+1}{1-t}}$$

around $t = 0$. This is the solution to the related homogeneous equation. The nonhomogeneous equation, then, we suspect has equation

$$x(t) = k(t)\sqrt{\frac{t+1}{1-t}}$$

We want to check when

$$\frac{dx}{dt} + \frac{x}{t^2-1} = \frac{2}{2t-1}$$

We have

$$\frac{dx}{dt} = k'(t)\sqrt{\frac{t+1}{1-t}} + k(t)\frac{d}{dt}\sqrt{\frac{t+1}{1-t}}$$

By the chain, power, and quotient rules, we have that

$$\frac{d}{dt}\sqrt{\frac{t+1}{1-t}} = \frac{1}{2\sqrt{\frac{t+1}{1-t}}} \cdot \frac{1(1-t)-(-1)(t+1)}{(1-t)^2} = \frac{1}{\sqrt{(1-t)^3(1+t)}}$$

so

$$\frac{dx}{dt} + \frac{x}{t^2-1} = k'(t)\sqrt{\frac{t+1}{1-t}} + \frac{k(t)}{\sqrt{(1-t)^3(1+t)}} + \frac{k(t)\sqrt{\frac{t+1}{1-t}}}{t^2-1}$$

The second two terms cancel out because

$$\frac{\sqrt{\frac{t+1}{1-t}}}{t^2-1} = -\frac{\sqrt{t+1}}{\sqrt{1-t}(1-t)(1+t)} = -\frac{1}{\sqrt{(1-t)^3(1+t)}}$$

so we have

$$k'(t)\sqrt{\frac{t+1}{1-t}} = \frac{2}{2t-1}$$

or

$$k'(t) = \frac{2\sqrt{1-t}}{\sqrt{t+1}(2t-1)}$$

The solution, $k(t)$, is not a function that is easy to calculate. A solution for $k(t)$ exists, however, so then we have that the solution is

$$x(t) = k(t)\sqrt{\frac{t+1}{1-t}}$$

This is the use of the existence theorem: we are told for sure that a solution exists even if we are not eventually able to find it. Factors in the form of the solution also serve to indicate discontinuities in the solution, corroborating the results of the existence-uniqueness theorem. We suspect from the form of $k'(t)$ that the solution will not be defined at $t = \frac{1}{2}$ or $t = \pm 1$. Then, a unique solution to the initial value problem $x(0) = 0$ will exist on the interval between -1 and $\frac{1}{2}$. The solution, however, even if it is defined outside of this range, could vary because the constants could be different outside of this range. In other words, when we find $k(t)$ there will be a constant of integration, but this constant could be different on any of the different ranges on which the solution is potentially defined. Then, the solution is not unique outside of the small range between -1 and $\frac{1}{2}$. Because of the discontinuities, we could have different constants for different sections.

The logic is exactly the same if the initial condition is $x(2) = 0$. All of the same arguments apply: there exists a unique solution function, but the interval of validity for uniqueness is

$$\boxed{\textbf{If } x(2) = 0\textbf{, there exists a unique solution on } 1 < t \leq \infty}$$

9. $\frac{dx}{dt} + x\tan t = 1$ with $x(3) = 5$. What if instead of $x(3) = 5$ we have $x(5) = 5000$?

This problem has a new twist. It is in the correct form,

$$\frac{dx}{dt} + p(t)x = q(t)$$

with

$$p(t) = \tan t, q(t) = 1$$

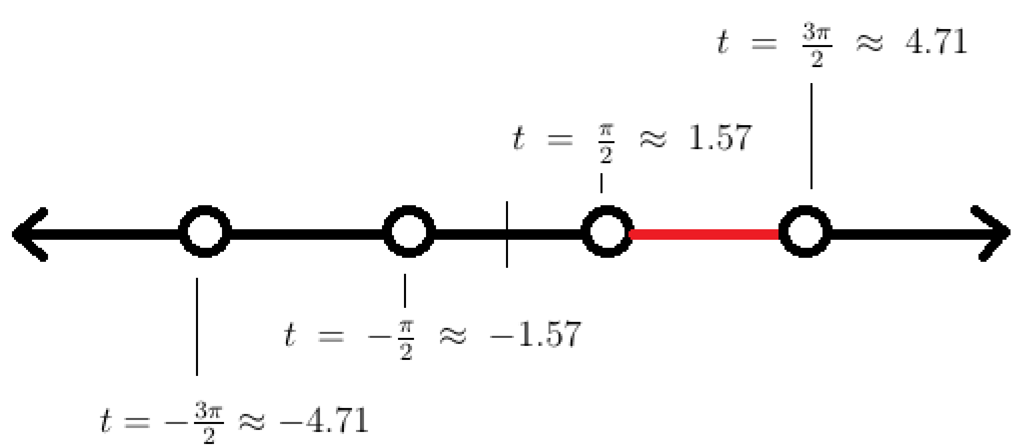
The function $q(t)$ is continuous everywhere, but $p(t)$ is more complicated. Rather than having a point or section of discontinuity, or several points of discontinuity, this function $p(t)$ is discontinuous at infinitely many points. Since

$$\tan t = \frac{\sin t}{\cos t}$$

it is discontinuous wherever $\cos t = 0$ - that is, at

$$\frac{\pi}{2} + \pi n$$

for any integer n. Given the initial condition $x(3) = 5$, then, the existence-uniqueness theorem indicates that there exists a solution on the interval where $p(t)$ and $q(t)$ are continuous that includes $t = 3$:



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This is the red interval between

$$\frac{\pi}{2} < t < \frac{3\pi}{2}$$

so

$$\boxed{\textbf{With } x(3) = 5\textbf{, there exists a unique solution on } \frac{\pi}{2} < t < \frac{3\pi}{2}}$$

For $x(5) = 5000$, $t = 5$ is more than $\frac{3\pi}{2} \approx 4.71$ so it is not in the same red interval, but t is clearly smaller than $\frac{5\pi}{2}$, so

$$\boxed{\textbf{With } x(5) = 5000\textbf{, there exists a unique solution on } \frac{3\pi}{2} < t < \frac{5\pi}{2}}$$

Note that the value of x does not matter in these interval of validity problems (even if it is very large), only the value of t.

The meaning of all these discontinuities is that a solution to the differential equation could potentially have a different solution in each interval. The initial value only dictates one solution over the interval where the initial value is, but the other solutions are free to vary. Therefore the interval of uniqueness is only the interval of continuity where the initial value is. To understand this, we can solve the differential equation. It is a nonhomogeneous linear equation, so the related homogeneous equation is

$$\frac{dx}{dt} = -x \tan t$$

Then,

$$\frac{1}{x} dx = -\tan t dt$$

so

$$\ln |x| = -\int \tan t dt = (\ln |\cos t|) + C$$

Then,

$$|x| = e^{(\ln |\cos t|)+C} = C|\cos t|$$

On any interval, then,

$$x = C \cos t$$

Due to the absolute value, it is possible that on some intervals C is negative, but on other intervals C is positive. We know that x is never zero because if $\cos t = 0$ then the differential equation is not defined because $\tan t$ is not defined. This is the solution to the related homogeneous equation. We expect a solution to the nonhomogeneous equation of the form

$$x(t) = k(t)\cos t$$

Then,

$$\frac{dx}{dt} = k'(t)\cos t - k(t)\sin t$$

and

$$\frac{dx}{dt} + x\tan t = k'(t)\cos t - k(t)\sin t + k(t)\sin t = k'(t)\cos t$$

which we set equal to 1, so

$$k'(t) = \sec t$$

and

$$k(t) = (\ln|\tan t + \sec t|) + C$$

as a table of integrals will indicate. Then,

$$x(t) = \cos t\,(\ln|\tan t + \sec t|) + C\cos t$$

This is defined everywhere except at $\frac{\pi}{2} \pm \pi n$. If $x(3) = 5$, there will be a solution for C that is valid between the two adjacent zeroes of cosine. But outside of these two zeroes, C could be something different. We could have C_1 defined for t between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$, then C_2 defined for t between $\frac{5\pi}{2}$ and $\frac{7\pi}{2}$, and so on. This does not create a continuity issue because the solution cannot be defined at $\frac{\pi}{2} + n\pi$ where the two different solution functions would meet. The solution is only unique in the interval where the initial condition is given, because there the constant is fixed. If $x(5) = 5000$, there is a solution for C in the interval between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$. The fact that $x = 5000$ does not affect the existence of an answer; it just means that C must be large. But the constants in other intervals are not prescribed, so the interval of uniqueness again has length π.

10. $\frac{dy}{dx} + y \csc \frac{x}{2} = x$ with $y(-3) = 4$. What if instead of $y(-3) = 4$ we have $y(3) = 4$?

The logic in this problem is the same as in the last problem. We have

$$p(x) = \csc \frac{x}{2}$$

and

$$q(x) = x$$

All of the continuity issues are with $p(x)$, since $q(x)$ is continuous everywhere. There is a discontinuity in $p(x)$ whenever

$$\sin \frac{x}{2} = 0$$

since $\csc x = \frac{1}{\sin x}$. This occurs whenever

$$\frac{x}{2} = \pi n$$

or

$$x = 2\pi n$$

Then, the intervals of validity are between $2\pi(n-1)$ and $2\pi n.$ If $y(-3) = 4$, a solution exists and is unique between -2π and 0, since this is the range that -3 falls in. If $y(3) = 4$, a solution exists and is unique between 0 and 2π, since this is the range that 3 falls in.

$$\boxed{\textbf{With } y(-3) = 4\textbf{, there exists a unique solution on } -2\pi < t < 0}$$

and

$$\boxed{\textbf{With } y(3) = 4\textbf{, there exists a unique solution on } 0 < t < 2\pi}$$

We can assert this confidently by the existence-uniqueness theorem without having to actually calculate the solution. This is very fortunate since the solution is very complicated, with many different expressions of cotangent (as a calculator could reveal.)