Calculus 3 Tutor, Volume I

Worksheet 3

Dot Product

Worksheet for Calculus 3 Tutor, Volume I, Section 3:

Dot Product

1. Compute the following dot products:

   (a) $(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \cdot \left(\hat{\boldsymbol{k}}\right)$;

   (b) $\left(2\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}\right) \cdot \left(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right)$;

   (c) $\left(5\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}\right) \cdot \left(3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{k}}\right)$;

   (d) $\left(\sqrt{3}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \sqrt{2}\hat{\boldsymbol{k}}\right) \cdot \left(\sqrt{3}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \sqrt{2}\hat{\boldsymbol{k}}\right)$;

(e) $\left(-\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}\right) \cdot \left(\sqrt{2}\hat{\boldsymbol{\imath}} - \sqrt{2}\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}\right)$;

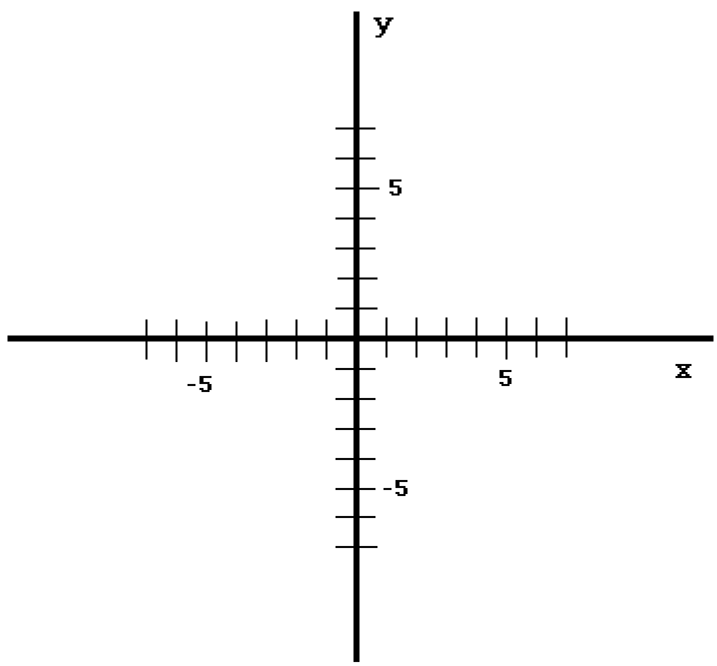
(f) $\left(\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) \cdot \left(-\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}\right)$;

(g) $\left(4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}\right) \cdot \left(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}\right)$;

(h) $\left(3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} - 5\hat{\boldsymbol{k}}\right) \cdot \left(3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right)$;

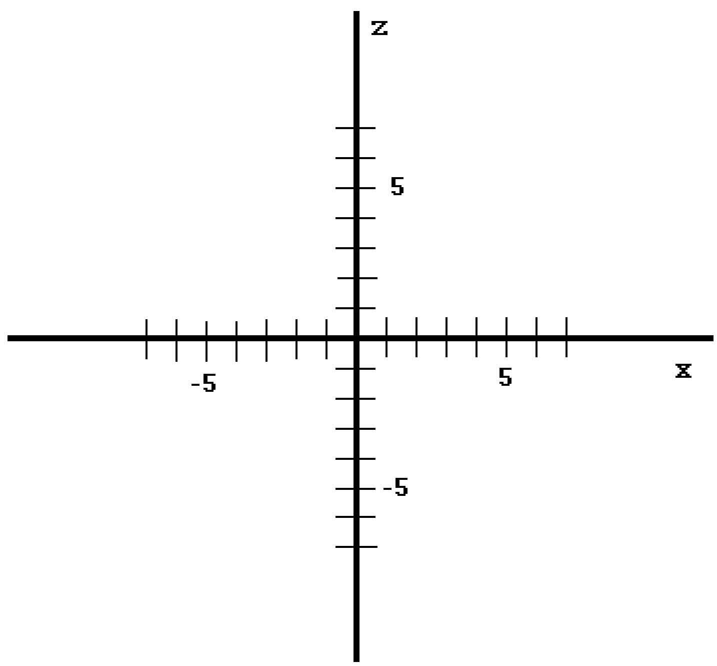
2. Find the angle between the following vectors using the dot product formula:

(a) $\vec{a} = 3\hat{\imath} - 3\hat{\jmath}$ and $\vec{b} = -3\hat{\imath} + 3\hat{\jmath}$. Plot these two vectors on the following plane to understand the result:



Plot for 2a

(b) $\vec{a} = 4\hat{\boldsymbol{\imath}}$ and $\vec{b} = 3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{k}}$. Plot these two vectors on the following plane (note that the axes are labeled x and z) to understand the result:



Plot for 2b

(c) $\vec{a} = \frac{1}{2}\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}} + \frac{\sqrt{2}}{2}\hat{\boldsymbol{k}}$; $\vec{b} = -\frac{1}{2}\hat{\boldsymbol{\imath}} - \frac{1}{2}\hat{\boldsymbol{\jmath}} + \frac{\sqrt{2}}{2}\hat{\boldsymbol{k}}$;

(d) $\vec{a} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$; $\vec{b} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$;

(e) $\vec{a} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$; $\vec{b} = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$. You may use a calculator to find the angle.

(f) $\vec{a} = 3\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$; $\vec{b} = 4\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$.You may use a calculator to find the angle.

(g) The unit vector in the direction of $\vec{a} = 7\hat{\imath} - 8\hat{\jmath} + 9\hat{k}$, and the vector $\vec{b} = -7\hat{\imath} - 9\hat{\jmath} - 8\hat{k}$. You may use a calculator to compute the inverse trigonometric function and find the angle.

(h) The unit vector in the direction of $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$, and the unit vector in the direction of $\vec{b} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$. You may use a calculator to compute the inverse trigonometric function and find the angle.

(i) Geometrically, what is the largest possible angle between two vectors? The smallest possible angle? Does the range of $\theta = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$ match the geometric result for the possible angles?

3. For each of the given vectors, find a vector that is perpendicular to it. There is no single right answer.

   (a) $\hat{\boldsymbol{k}}$;

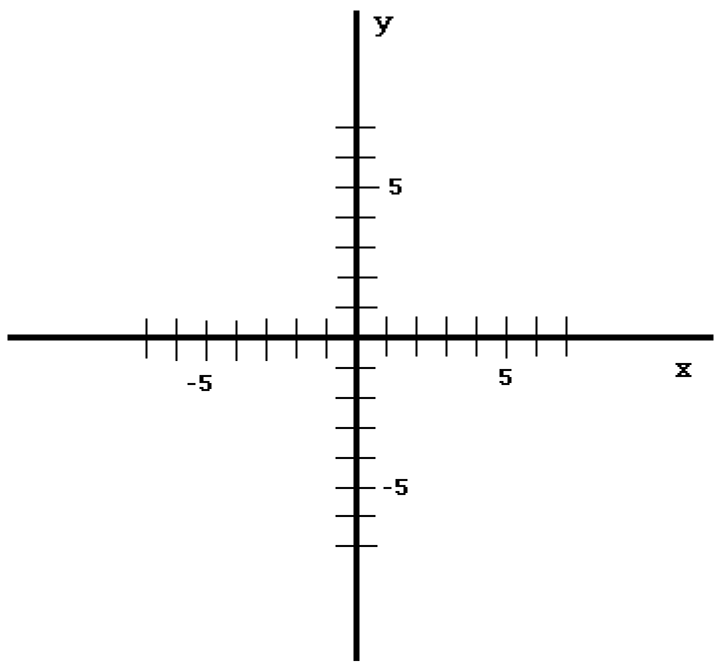
   (b) $2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}$;

   (c) $\left(\cos\frac{\pi}{6}\right)\hat{\boldsymbol{\imath}} + \left(\sin\frac{\pi}{6}\right)\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$;

   (d) both $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$.

4. Vectors at a given angle from $\hat{\imath}$:

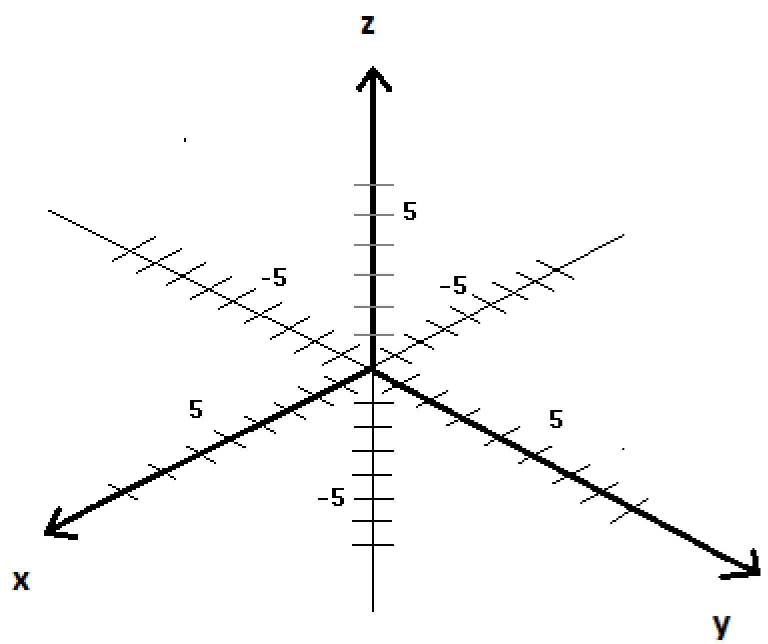
   (a) Plot a vector that is $\frac{\pi}{2}$ degrees from $\hat{\imath}$ (that is, perpendicular to $\hat{\imath}$) on the grid below. Algebraically, what conditions must a vector satisfy for it to be $\frac{\pi}{2}$ degrees from $\hat{\imath}$? Geometrically, what conditions must the vector satisfy?



Plot for 4a

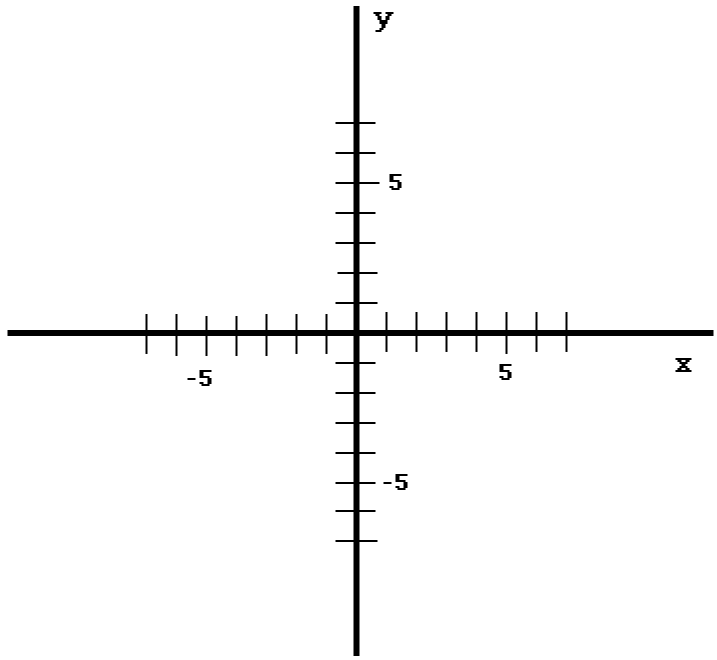
   (b) Algebraically, what conditions must a vector in the 3D plane satisfy for it to be $\frac{\pi}{2}$

degrees from $\hat{\imath}$? Indicate this set of vectors graphically on the plot:



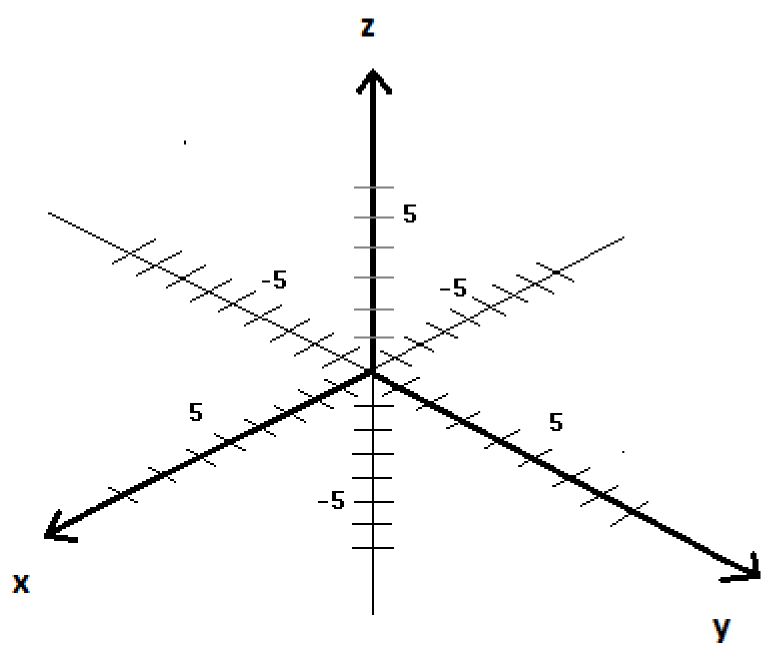
Plot for 4b

(c) Find the algebraic condition for a vector in the two-dimensional plane that is $\frac{\pi}{4}$ degrees from $\hat{\imath}$. Indicate this set of vectors graphically on the plot:



Plot for 4c

(d) **Challenge.** Find the algebraic condition for a vector in the three-dimensional plane that is $\frac{\pi}{4}$ degrees from $\hat{\imath}$. What shape do all these vectors form?



Plot for 4d

5. Prove the following identities regarding the dot product:

(a) $(c\vec{a}) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right)$;

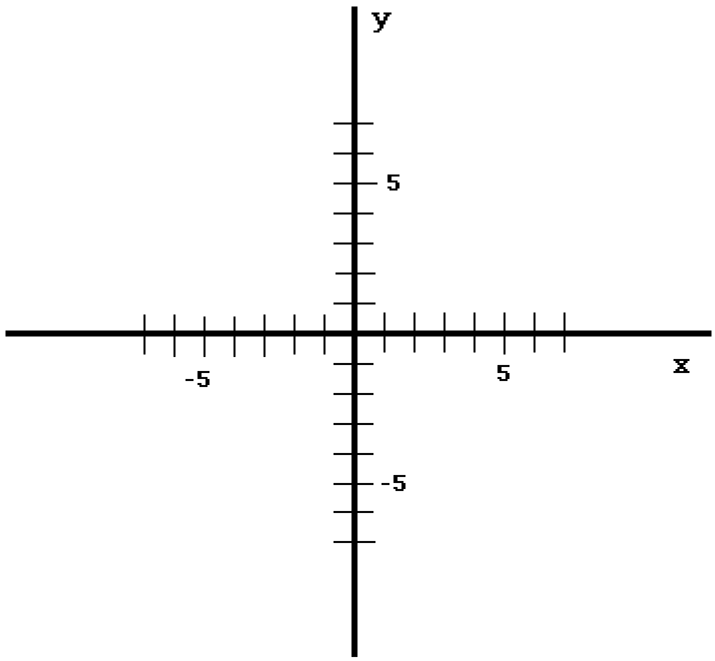
(b) $\left(\vec{a} + \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$;

(c) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$;

(d) $\vec{a} \cdot \vec{b} \leq \|\vec{a}\| \left\|\vec{b}\right\|$. Prove this identity using the algebraic definition $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

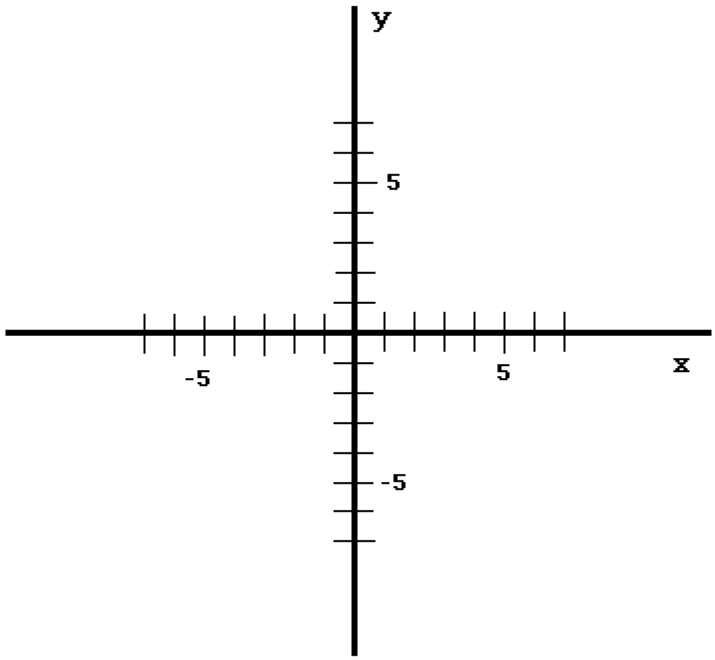
6. Working with the algebraic and trigonometric definitions of the dot product:

   (a) Prove algebraically (without using the dot product formula) that if $\vec{a}$ and $\vec{b}$ in the 2D plane are perpendicular, then $a_1 b_1 + a_2 b_2 = 0$. Hint: use the Pythagorean theorem. It may help to plot an example on the following plane.



Plot for 6a

(b) Prove that $a_1b_1 + a_2b_2 = \|a\|\,\|b\|\cos\theta$ in the two-dimensional plane. Hint: use the Law of Cosines, which indicates that in any triangle $c^2 = a^2 + b^2 - 2ab\cos\theta$. It may help to plot an example on the following plane.



Plot for 6b

(c) Prove that $a_1b_1 + a_2b_2 + a_3b_3 = \|a\| \, \|b\| \cos\theta$ in three-dimensional space.

Answer key.

1. Computing dot products:

1(a). **Answer:** $\boxed{0}$. The dot product of two vectors $\vec{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\vec{b} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$ is

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Here,

$$\vec{a} = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$$

and

$$\vec{b} = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$$

Then,

$$\vec{a} \cdot \vec{b} = (1 \cdot 0) + (2 \cdot 0) + (0 \cdot 2) = 0 + 0 + 0$$

which is $\boxed{0}$. The dot product multiplies corresponding components: it adds the product of the x-components, the y-components, and the z-components. Each of these products is zero. None of the components is nonzero in both $\vec{a}$ and $\vec{b}$. Therefore the dot product is zero.

1(b). **Answer:** $\boxed{-1}$. To compute the dot product, we use the formula

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Here, a_1, a_2, a_3 are the terms of $\vec{a}$, which are $0, 2, -3$. Likewise, b_1, b_2, b_3 are the terms of $\vec{b}$, which are $1, 1, 1$. Then, the product is

$$\vec{a} \cdot \vec{b} = 0 \cdot 1 + 2 \cdot 1 + -3 \cdot 1 = 0 + 2 - 3$$

which is $\boxed{-1}$.

1(c). **Answer:** $\boxed{1}$. The dot product of these two vectors is given by adding the product of their components. We have that

$$\vec{a} = 5\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} + 7\hat{\boldsymbol{k}}$$

and

$$\vec{b} = 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{k}} = 3\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$$

Then, the product of the x-components is $5 \cdot 3 = 15$. The product of the y-components is $-6 \cdot 0 = 0$. The product of the z-components is $7 \cdot -2 = -14$. Summing all of these products, we get

$$\vec{a} \cdot \vec{b} = 15 + 0 - 14$$

which is $\boxed{1}$.

1(d). **Answer:** $\boxed{0}$. To take the dot product

$$\vec{a}\cdot\vec{b} = \left(a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}\right)\cdot\left(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}\right)$$

we compute

$$a_1b_1 + a_2b_2 + a_3b_3$$

Then,

$$\left(\sqrt{3}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \sqrt{2}\hat{\boldsymbol{k}}\right)\cdot\left(\sqrt{3}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \sqrt{2}\hat{\boldsymbol{k}}\right) = \left(\sqrt{3}\right)\left(\sqrt{3}\right) + (-1)(1) + \left(\sqrt{2}\right)\left(-\sqrt{2}\right) = 3-1-2$$

which is $\boxed{0}$.

1(e). **Answer:** $\boxed{21}$. To take the dot product, we first multiply corresponding terms:

$$a_1b_1 = -1\cdot\sqrt{2} = -\sqrt{2}$$

$$a_2b_2 = -1\cdot\left(-\sqrt{2}\right) = \sqrt{2}$$

$$a_3b_3 = 7\cdot 3 = 21$$

Adding these three together yields the answer $\boxed{21}$.

1(f). **Answer:** $\boxed{0}$. We simplify

$$\left(\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) \cdot \left(-\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}\right) = \hat{\boldsymbol{\imath}} \cdot (-\hat{\boldsymbol{\imath}}) + 3\hat{\boldsymbol{\jmath}} \cdot (-3\hat{\boldsymbol{\jmath}}) + 5\hat{\boldsymbol{k}} \cdot 2\hat{\boldsymbol{k}}$$

because, after we distribute, all of the other terms cancel since $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} = 0$.

The first product is

$$\hat{\boldsymbol{\imath}} \cdot (-\hat{\boldsymbol{\imath}}) = -1$$

The second product is

$$3\hat{\boldsymbol{\jmath}} \cdot (-3\hat{\boldsymbol{\jmath}}) = -9$$

The third product is

$$5\hat{\boldsymbol{k}} \cdot 2\hat{\boldsymbol{k}} = 10$$

Adding these three together gives

$$-1 - 9 + 10 = \boxed{0}$$

1(g). **Answer:** $\boxed{-3}$. The dot product is given by

$$a_1b_1 + a_2b_2 + a_3b_3$$

Here,

$$a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}} = 4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}$$

and

$$b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}} = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$$

so the dot product is

$$(4)(2) + (-3)(1) + (8)(-1) = 8 - 3 - 8$$

which is $\boxed{-3}$.

1(h). **Answer:** $\boxed{0}$. The dot product is

$$\left(3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} - 5\hat{\boldsymbol{k}}\right) \cdot \left(3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\right) = 3^2 + 4^2 - 5^2$$

which is

$$9 + 16 - 25$$

which is equal to $\boxed{0}$.

2. Finding the angle between vectors:

2(a). **Answer:** $\boxed{\pi}$. We use the formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. The terms $\vec{a} \cdot \vec{b}$, $\|\vec{a}\|$, $\left\|\vec{b}\right\|$ can be determined algebraically, and then it will be possible to determine $\cos\theta$. We have

$$\vec{a} \cdot \vec{b} = (3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}) \cdot (-3\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}) = 3 \cdot (-3) + (-3) \cdot 3 = -18$$

We can also compute

$$\|\vec{a}\| = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

and

$$\left\|\vec{b}\right\| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

Then,

$$-18 = \left(3\sqrt{2}\right)\left(3\sqrt{2}\right)\cos\theta$$

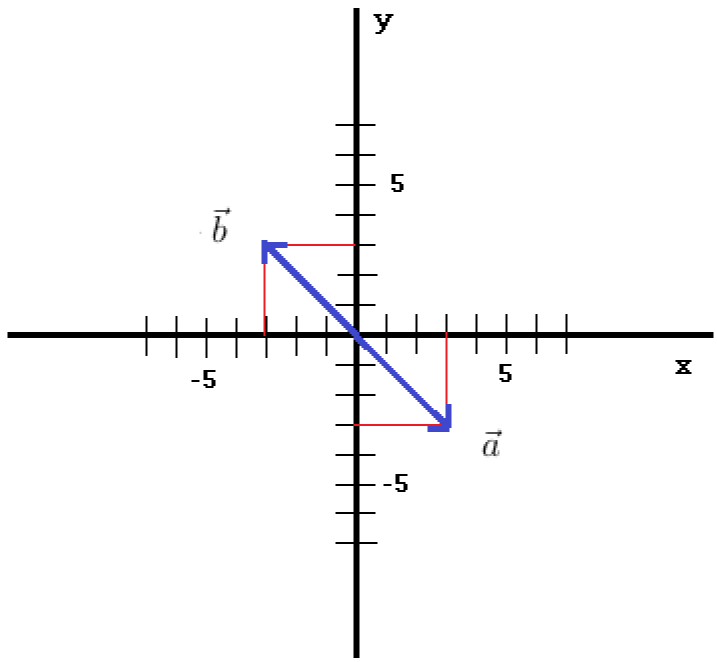
which reduces to

$$-18 = 18\cos\theta$$

or

$$\cos\theta = -1$$

Then, the solution is $\theta = \boxed{\pi}$. The vectors $\vec{a}$ and $\vec{b}$ have an angle π - that is, a straight angle - between them. This is concordant with the plot of these vectors (below):



Plot for 2a

The vectors are clearly pointing in opposite directions, with an angle measuring π radians or 180 degrees between them. This confirms the result that we found using the dot product formula.

2(b). **Answer:** $\boxed{\dfrac{\pi}{4}}$. We use the formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$$

where θ is the angle between them. Here,

$$\vec{a} \cdot \vec{b} = 4\hat{\boldsymbol{\imath}} \cdot \left(3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{k}}\right) = 12$$

We can also simply compute

$$\|\vec{a}\| = \sqrt{4^2} = 4$$

and

$$\left\|\vec{b}\right\| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$$

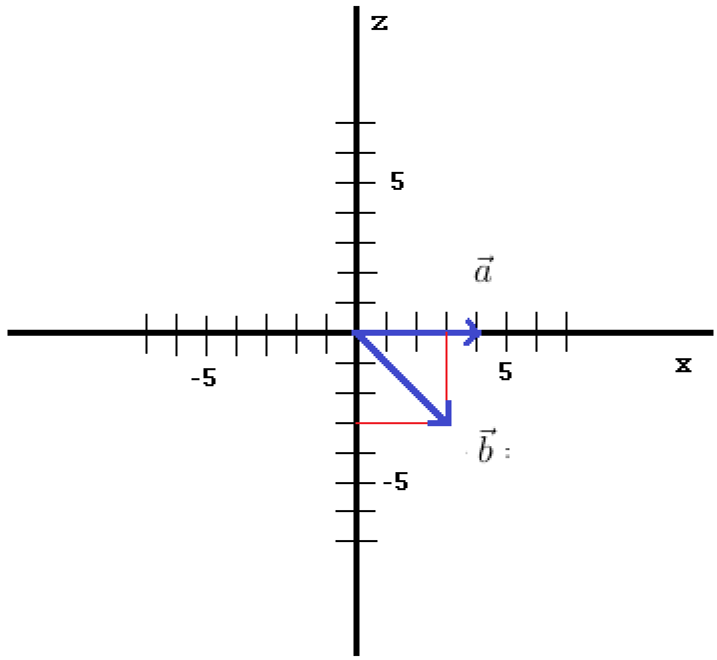
Then,

$$12 = 4 \cdot 3\sqrt{2} \cdot \cos\theta$$

so

$$\cos\theta = \frac{1}{\sqrt{2}}$$

which means $\theta = \boxed{\dfrac{\pi}{4}}$. To show this result graphically, we can plot these vectors in the xz plane. Whenever we compute the angle between two vectors, we are implicitly drawing a plane that includes them both, and calculating the angle in that plane (as we could measure an angle in the xy plane). Here, the plane is the xz plane where y is zero. The plot is as follows:



Plot for 2b

When the vectors are drawn in the correct plane, it is clear that the angle between them is indeed $\frac{\pi}{4}$, confirming the result we found by taking the dot product.

2(c). **Answer:** $\boxed{\dfrac{\pi}{2}}$. We use the formula

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\left\|\vec{b}\right\|}$$

Here,

$$\vec{a}\cdot\vec{b} = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) + \frac{1}{2}\cdot\left(-\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{1}{4} - \frac{1}{4} + \frac{1}{2}$$

which is zero. It isn't even necessary to calculate the magnitudes of $\vec{a}$ and $\vec{b}$, since - regardless of the answer -

$$\cos\theta = 0$$

or $\boxed{\theta = \dfrac{\pi}{2}}$. These vectors are perpendicular.

2(d). **Answer:** $\boxed{\frac{\pi}{3}}$. We use the formula

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\left\|\vec{b}\right\|}$$

We compute

$$(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}})\cdot\left(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{k}}\right) = 1$$

Also,

$$\|\vec{a}\| = \left\|\vec{b}\right\| = \sqrt{1^2+1^2} = \sqrt{2}$$

Then,

$$\cos\theta = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$$

which means that $\boxed{\theta = \frac{\pi}{3}}$ There would be no way to estimate this angle graphically; the dot product formula allows us to compute this angle exactly.

2(e). **Answer:** $\boxed{\cos^{-1}\frac{1}{3} \approx 1.23}$. The hint to use the calculator indicates that the angle will not have a simple result as a fraction of π. Even if the angle is not simple, the vectors can be expressed simply. We use the formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$$

where θ is the angle between these two vectors. Here

$$\vec{a} \cdot \vec{b} = 1^2 - 1^2 + 1^2 = 1$$

Also, we can compute

$$\|\vec{a}\| = \left\|\vec{b}\right\| = \sqrt{3}$$

Then,

$$1 = \left(\sqrt{3}\right)^2 \cos\theta$$

which means

$$\cos\theta = \frac{1}{3}$$

or

$$\theta = \boxed{\cos^{-1}\frac{1}{3} \approx 1.23}$$

where the numerical answer is found using a calculator.

2(f). **Answer:** $\boxed{\cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76}$. Again we use the formula

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\left\|\vec{b}\right\|}\right)$$

Here,

$$\vec{a}\cdot\vec{b} = 3\cdot 4 + (-6)\cdot 4 + 2\cdot 2 = 12 - 24 + 4 = -8$$

The norms of both vectors are integers:

$$\|\vec{a}\| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

$$\left\|\vec{b}\right\| = \sqrt{4^2 + 4^2 + 2^2} = \sqrt{16 + 16 + 4} = \sqrt{36} = 6$$

Then,

$$\theta = \cos^{-1}\frac{-8}{6\cdot 7}$$

which gives

$$\theta = \boxed{\cos^{-1}\left(-\frac{4}{21}\right) \approx 1.76}$$

2(g). **Answer:** $\boxed{\cos^{-1}\left(-\frac{49}{194}\right) \approx 1.82}$. This seems like a more complicated problem because it asks for the unit vector. However, the unit vector is different from the original vector only in direction, not in angle. Therefore the angle between $\vec{a}$ and $\vec{b}$ is the same as the angle between the unit vector in the direction of $\vec{a}$, and $\vec{b}$. To prove this algebraically, we write

$$\hat{a} = \frac{\vec{a}}{\|a\|}$$

as the unit vector in the direction of $\vec{a}$. Then, the angle between $\vec{a}$ and $\vec{b}$ is

$$\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,\left\|\vec{b}\right\|}\right)$$

and the angle between $\hat{a}$ and $\vec{b}$ is

$$\cos^{-1}\left(\frac{\hat{a}\cdot\vec{b}}{\|\hat{a}\|\,\left\|\vec{b}\right\|}\right)$$

By definition,

$$\|\hat{a}\| = 1$$

Then,

$$\cos^{-1}\left(\frac{\hat{a}\cdot\vec{b}}{\|\hat{a}\|\,\left\|\vec{b}\right\|}\right) = \cos^{-1}\left(\frac{\left(\frac{\vec{a}}{\|a\|}\right)\cdot\vec{b}}{1\cdot\left\|\vec{b}\right\|}\right)$$

which is the same as the angle between $\vec{a}$ and $\vec{b}$. So to find the angle between the unit vector in the direction of $\vec{a} = 7\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 9\hat{\boldsymbol{k}}$ and the vector $\vec{b} = -7\hat{\boldsymbol{\imath}} - 9\hat{\boldsymbol{\jmath}} - 8\hat{\boldsymbol{k}}$ (that is, the angle between $\hat{a}$ and $\vec{b}$), we can simply find the angle between $7\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} + 9\hat{\boldsymbol{k}}$ and $-7\hat{\boldsymbol{\imath}} - 9\hat{\boldsymbol{\jmath}} - 8\hat{\boldsymbol{k}}$. This is

$$\theta = \cos^{-1}\left(\frac{-49+72-72}{\left(\sqrt{7^2+8^2+9^2}\right)^2}\right) = \cos^{-1}\left(-\frac{49}{194}\right)$$

The angle is $\boxed{\cos^{-1}\left(-\frac{49}{194}\right) \approx 1.82}$.

2(h). **Answer:** $\boxed{\cos^{-1}\left(-\frac{24}{49}\right) \approx 2.08}$. As above, we do not need to actually compute the unit vectors to find the angle, since the angle between unit vectors is the same as the angle between the original vectors. This angle is

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\left\|\vec{b}\right\|}\right)$$

Here,

$$\vec{a}\cdot\vec{b} = 6 - 18 - 12 = -24$$

and

$$\|\vec{a}\|\left\|\vec{b}\right\| = \sqrt{3^2+6^2+2^2} = \sqrt{9+36+4} = \sqrt{49} = 7$$

Then,

$$\theta = \cos^{-1}\left(-\frac{24}{7\cdot 7}\right)$$

which gives an answer of

$$\theta = \boxed{\cos^{-1}\left(-\frac{24}{49}\right) \approx 2.08}$$

2(i). **Answer:** $\boxed{0 \leq \theta \leq \pi}$. Of course, the angle between two vectors can't be any less than zero. It can be zero if the two vectors are the same direction. Also, the angle can't be any greater than π. If the angle θ were greater than π, we would be able to measure the converse angle to get $2\pi - \theta$, which would be less than π and would be the correct angle between the two vectors. The angle is exactly π if the vectors are opposite. Therefore the range of angles between two vectors is $\boxed{0 \leq \theta \leq \pi}$. This is the same result as

$$\theta = \cos^{-1} \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \left\|\vec{b}\right\|}$$

because the range of $\cos^{-1} x$ is also

$$0 \leq \cos^{-1} x \leq \pi$$

.

3. Finding perpendicular vectors:

3(a). **Answer:** $\boxed{\hat{\boldsymbol{\imath}}}$ or $\boxed{\hat{\boldsymbol{\jmath}}}$ are the simplest answers, although any answer of the form $\boxed{a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}}$ is correct. If vectors are perpendicular, then the angle between them is $\frac{\pi}{2}$. If $\theta = \frac{\pi}{2}$, then the dot product formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$$

simplifies because $\cos\frac{\pi}{2} = 0$. That is, if $\vec{a}$ and $\vec{b}$ are perpendicular,

$$\vec{a} \cdot \vec{b} = 0$$

Here, $\vec{a} = \hat{\boldsymbol{k}}$ and so we want to find a vector $\vec{b}$ so that

$$\hat{\boldsymbol{k}} \cdot \vec{b} = 0$$

The vector $\boxed{\hat{\boldsymbol{\imath}}}$ dots to zero with $\hat{\boldsymbol{k}}$, as does the vector $\boxed{\hat{\boldsymbol{\jmath}}}$. In fact, any vector $\boxed{a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}}$ dots to zero with $\hat{\boldsymbol{k}}$, since

$$\hat{\boldsymbol{k}} \cdot (a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) = a\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}}\right) + b\left(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\jmath}}\right) = 0 + 0$$

This means that any vector $\boxed{a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}}$ is perpendicular to $\hat{\boldsymbol{k}}$.

3(b). **Answer:** $\boxed{2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}}$ although there are many other equally acceptable answers. We want to find $\vec{b}$ so that

$$\left(2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}\right) \cdot \vec{b} = 0$$

because this would mean that $2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - 6\hat{\boldsymbol{k}}$ is perpendicular to $\vec{b}$. That is, we want to find b_1, b_2, b_3 so that

$$2b_1 + 3b_2 - 6b_3 = 0$$

There are a multitude of possible solutions. One solution is found by taking $b_1 = 0, b_2 = 2, b_3 = 1$ which yields the perpendicular vector $\boxed{2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}}$, although there are many other correct answers. Specifying the general form of the answer is a topic in linear algebra.

3(c). **Answer:** $\boxed{\frac{\sqrt{3}}{2}\hat{\boldsymbol{\imath}}+\frac{1}{2}\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}}$ although there are many other equally acceptable answers. The questions asks us to find $\vec{b}$ so that

$$\left(\left(\cos\frac{\pi}{6}\right)\hat{\boldsymbol{\imath}}+\left(\sin\frac{\pi}{6}\right)\hat{\boldsymbol{\jmath}}-\hat{\boldsymbol{k}}\right)\cdot\left(b_1\hat{\boldsymbol{\imath}}+b_2\hat{\boldsymbol{\jmath}}+b_3\hat{\boldsymbol{k}}\right)=0$$

or

$$b_1\cos\frac{\pi}{6}+b_2\sin\frac{\pi}{6}-b_3=0$$

We could expand the cosines and sines, but it makes it simpler if we note that

$$\cos^2\frac{\pi}{6}+\sin^2\frac{\pi}{6}=1$$

Then,

$$b_1=\cos\frac{\pi}{6}, b_2=\sin\frac{\pi}{6}, b_3=1$$

satisfies this equation. The vector

$$\vec{b}=\cos\frac{\pi}{6}\hat{\boldsymbol{\imath}}+\sin\frac{\pi}{6}\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}$$

is perpendicular to the original vector $\cos\frac{\pi}{6}\hat{\boldsymbol{\imath}}+\sin\frac{\pi}{6}\hat{\boldsymbol{\jmath}}-\hat{\boldsymbol{k}}$. We expand the trigonometric expressions in $\vec{b}$ to obtain one possible answer, $\boxed{\frac{\sqrt{3}}{2}\hat{\boldsymbol{\imath}}+\frac{1}{2}\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}}$.

3(d). **Answer:** $\boxed{\hat{\boldsymbol{k}}}$ although any multiple of $\hat{\boldsymbol{k}}$ is an equally acceptable answer. Only $\hat{\boldsymbol{k}}$ or some scalar multiple of it satisfies $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\jmath}} = 0$. We want to find a vector $\vec{b} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$ that is perpendicular to $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. For $\vec{b}$ to be perpendicular to $\hat{\boldsymbol{\imath}}$, we need

$$b_1 = 0$$

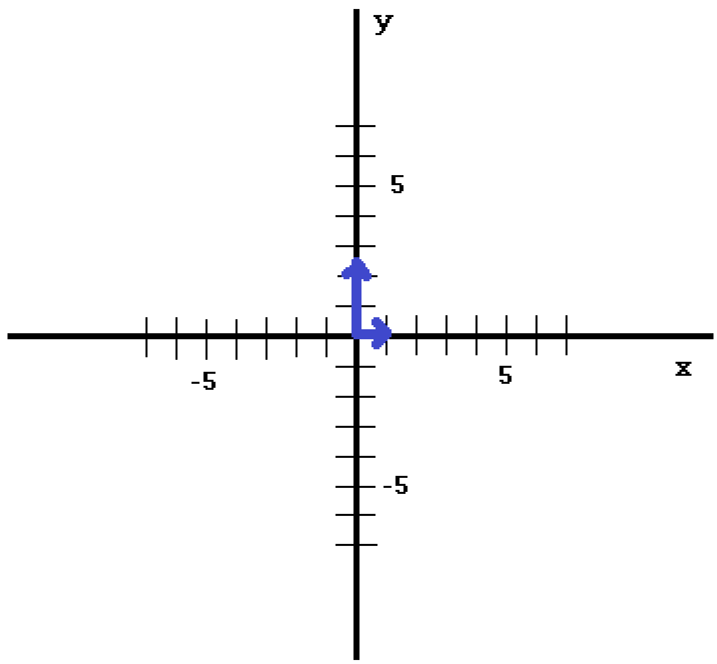
For $\vec{b}$ to be perpendicular to $\hat{\boldsymbol{\jmath}}$, we need

$$b_2 = 0$$

Therefore, the only vectors perpendicular to $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are multiples of $\hat{\boldsymbol{k}}$; and particularly $\boxed{\hat{\boldsymbol{k}}}$.

4. Finding vectors at a given angle from $\hat{\imath}$:

4(a). **Answer:** $\boxed{b\hat{\jmath}}$, some multiple of the unit vector $\hat{\jmath}$. The only vectors in the two-dimensional plane that are a right angle away from $\hat{\imath}$ are scalar multiples of $\hat{\jmath}$. This is shown in the following diagram:



Plot for 4a

Algebraically, then, a vector must be of the form $\boxed{b\hat{\jmath}}$ to be perpendicular to $\hat{\imath}$ in the two-dimensional plane. The dot product $(a\hat{\imath} + b\hat{\jmath}) \cdot \hat{\imath} = a$ is zero, indicating perpendicularity, only when $a = 0$. Note that b can be positive or negative; that is, the vector $b\hat{\jmath}$ can point up or down.

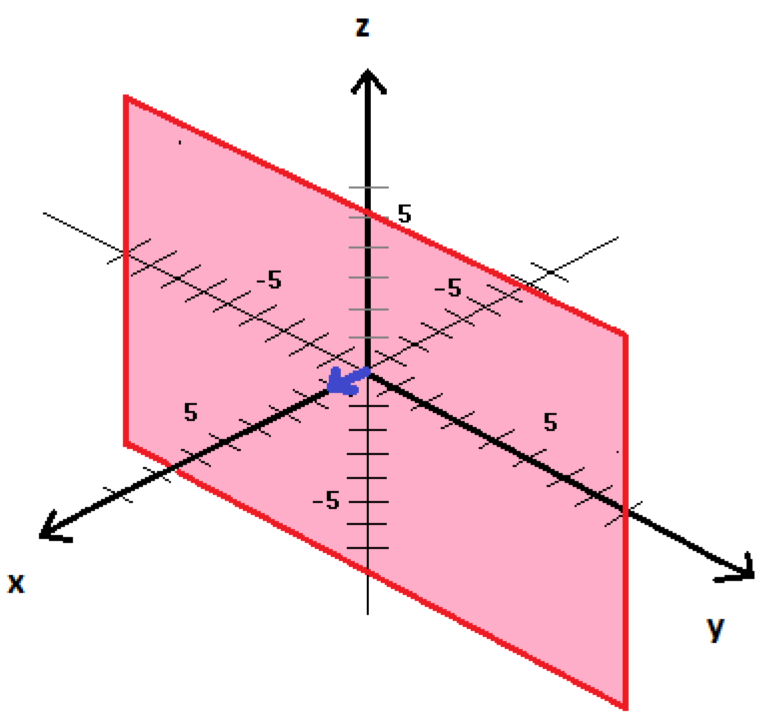
4(b). **Answer:** $\boxed{b\hat{\jmath} + c\hat{k}}$. For a vector in the three-dimensional plane to be perpendicular to $\hat{\imath}$, we need

$$\left(a\hat{\imath} + b\hat{\jmath} + c\hat{k}\right) \cdot \hat{\imath} = 0$$

or

$$a = 0$$

Any vector with x-component equal to zero - that is, any vector in the red plane - is perpendicular to $\hat{\imath}$. The algebraic result matches the graphical appearance. The pink plane is indeed perpendicular to the blue vector.



Plot for 4b

4(c). **Answer:** $\boxed{a\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}} \textbf{ and } a\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}; a > 0}$. A vector $\vec{a}$ is $\frac{\pi}{4}$ degrees from $\hat{\boldsymbol{\imath}}$ if

$$\vec{a} \cdot \hat{\boldsymbol{\imath}} = \|a\| \, \|\hat{\boldsymbol{\imath}}\| \cos \frac{\pi}{4}$$

If $\vec{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}}$ this means that

$$a_1 = \sqrt{a_1^2 + a_2^2} \cos \frac{\pi}{4}$$

Squaring both sides, this means that

$$a_1^2 = \left(a_1^2 + a_2^2\right) \cdot \frac{1}{2}$$

or

$$a_1^2 = a_2^2$$

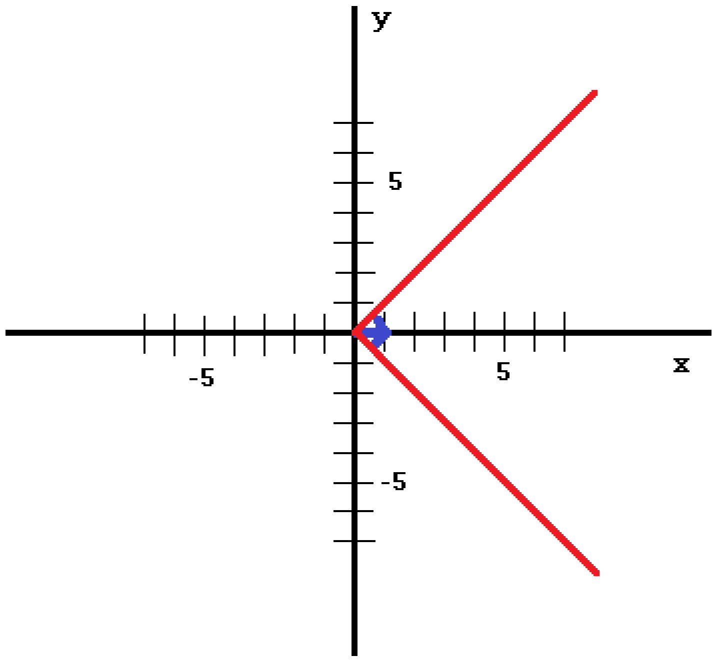
which means

$$a_1 = a_2$$

or

$$a_1 = -a_2$$

We have that a_1 must be positive because of the condition $a_1 = \sqrt{a_1^2 + a_2^2} \cos \frac{\pi}{4}$. Here, the right-hand side is positive, so a_1 must be positive. However, a_2 can be negative or positive. Therefore the vectors at an angle of $\frac{\pi}{4}$ from $\imath$ are $\boxed{a\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}} \textbf{ and } a\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}}$ for $a > 0$. The envelope of these vectors can be plotted as below.



Plot for 4c

Any vector along the red lines will be at an angle of $\frac{\pi}{4}$ from the vector $\hat{\imath}$, which is plotted in blue.

4(d). **Answer:** $\boxed{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \pm \left(\sqrt{x^2 - y^2}\right)\hat{\boldsymbol{k}}; x > 0; -x \le y \le x}$ which can also be expressed as $\boxed{\left(\sqrt{y^2 + z^2}\right)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}$ for any y, z. This is a $\boxed{\textbf{cone}}$. A vector $\vec{a}$ is $\frac{\pi}{4}$ from $\hat{\boldsymbol{\imath}}$ if

$$\vec{a} \cdot \hat{\boldsymbol{\imath}} = \|a\| \, \|\hat{\boldsymbol{\imath}}\| \cos \frac{\pi}{4}$$

We write

$$\vec{a} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

Then,

$$\vec{a} \cdot \hat{\boldsymbol{\imath}} = x$$

$$\|a\| = \sqrt{x^2 + y^2 + z^2}$$

$$\|\hat{\boldsymbol{\imath}}\| = 1$$

so

$$x = \sqrt{x^2 + y^2 + z^2} \cdot \frac{1}{\sqrt{2}}$$

Multiplying by $\sqrt{2}$ and then squaring both sides, we get

$$2x^2 = x^2 + y^2 + z^2$$

or

$$x^2 = y^2 + z^2$$

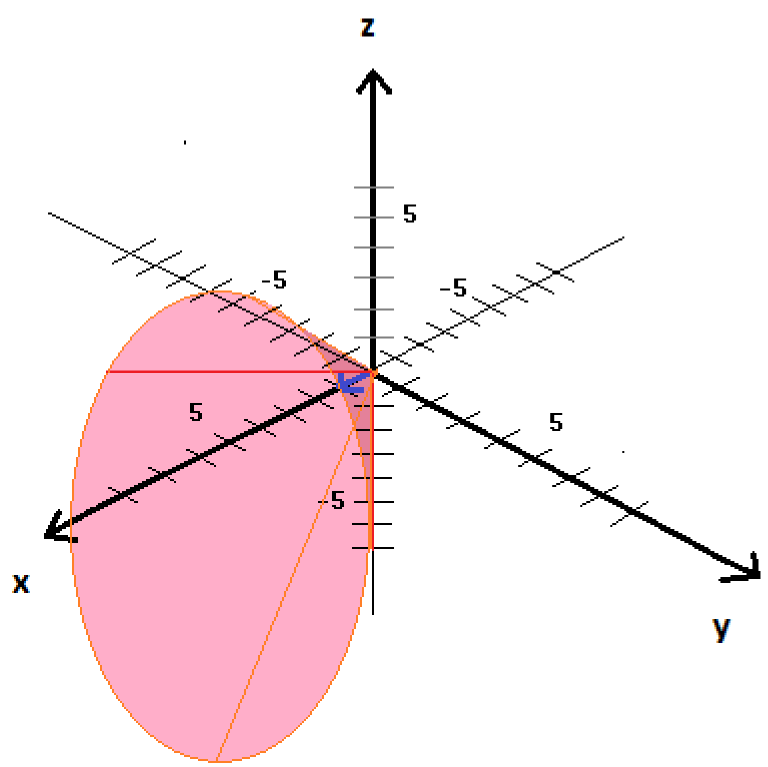
We see that x must be positive, since $x = \sqrt{x^2 + y^2 + z^2} \cdot \frac{1}{\sqrt{2}}$, but y and z can be positive or negative. Then, two expressions of the envelope of possible vectors are $\boxed{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \pm \left(\sqrt{x^2 - y^2}\right)\hat{\boldsymbol{k}}; x > 0; -x \le y \le x}$ since

$$z = \sqrt{x^2 - y^2}$$

or $\boxed{\left(\sqrt{y^2 + z^2}\right)\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}$ since

$$x = \sqrt{y^2 + z^2}$$

The algebraic expression does not allow us to easily plot this family of vectors that is at an angle $\frac{\pi}{4}$ from $\hat{\boldsymbol{\imath}}$. However, geometric thinking - or extrapolating from the 2D case - allows us to infer that this family of vectors at an angle of $\frac{\pi}{4}$ from $\hat{\imath}$ is a $\boxed{\textbf{cone}}$ around $\hat{\imath}$. The plot is as follows:



Plot for 4d

5. Proving dot product identities:

5(a). **Answer:** $\boxed{(c\vec{a}) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right)}$. We write

$$\vec{a} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$$

$$\vec{b} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$$

Then,

$$c\vec{a} = ca_1\hat{\boldsymbol{\imath}} + ca_2\hat{\boldsymbol{\jmath}} + ca_3\hat{\boldsymbol{k}}$$

and

$$(c\vec{a}) \cdot \vec{b} = \left(ca_1\hat{\boldsymbol{\imath}} + ca_2\hat{\boldsymbol{\jmath}} + ca_3\hat{\boldsymbol{k}}\right) \cdot \left(b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}\right) = ca_1b_1 + ca_2b_2 + ca_3b_3$$

We can factor this expression as

$$(c\vec{a}) \cdot \vec{b} = c\left(a_1b_1 + a_2b_2 + a_3b_3\right)$$

which we can write as $\boxed{(c\vec{a}) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right)}$, the identity we sought to prove.

5(b). **Answer:** $\boxed{\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}}$. We write

$$\vec{a}=a_1\hat{\boldsymbol{\imath}}+a_2\hat{\boldsymbol{\jmath}}+a_3\hat{\boldsymbol{k}}$$

$$\vec{b}=b_1\hat{\boldsymbol{\imath}}+b_2\hat{\boldsymbol{\jmath}}+b_3\hat{\boldsymbol{k}}$$

$$\vec{c}=c_1\hat{\boldsymbol{\imath}}+c_2\hat{\boldsymbol{\jmath}}+c_3\hat{\boldsymbol{k}}$$

Then,

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=\left((a_1+b_1)\,\hat{\boldsymbol{\imath}}+(a_2+b_2)\,\hat{\boldsymbol{\jmath}}+(a_3+b_3)\,\hat{\boldsymbol{k}}\right)\cdot\left(c_1\hat{\boldsymbol{\imath}}+c_2\hat{\boldsymbol{\jmath}}+c_3\hat{\boldsymbol{k}}\right)$$

which evaluates to

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=(a_1+b_1)\,(c_1)+(a_2+b_2)\,(c_2)+(a_3+b_3)\,(c_3)$$

Distributing, this is

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=(a_1c_1+b_1c_1)+(a_2c_2+b_2c_2)+(a_3c_3+b_3c_3)$$

Re-ordering the terms of the right-hand side, this is

$$\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=(a_1c_1+a_2c_2+a_3c_3)+(b_1c_1+b_2c_2+b_3c_3)$$

which we can identify as $\boxed{\left(\vec{a}+\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}}$, which is the result we set out to prove.

5(c). **Answer:** $\boxed{\vec{a}\cdot\vec{a}=\|\vec{a}\|^2}$. We write

$$\vec{a}=a_1\hat{\boldsymbol{\imath}}+a_2\hat{\boldsymbol{\jmath}}+a_3\hat{\boldsymbol{k}}$$

Then,

$$\vec{a}\cdot\vec{a}=\left(a_1\hat{\boldsymbol{\imath}}+a_2\hat{\boldsymbol{\jmath}}+a_3\hat{\boldsymbol{k}}\right)\cdot\left(a_1\hat{\boldsymbol{\imath}}+a_2\hat{\boldsymbol{\jmath}}+a_3\hat{\boldsymbol{k}}\right)=a_1^2+a_2^2+a_3^2$$

which is the same as $\|\vec{a}\|^2$ since

$$\|\vec{a}\|=\sqrt{a_1^2+a_2^2+a_3^2}$$

Then, $\boxed{\vec{a}\cdot\vec{a}=\|\vec{a}\|^2}$.

5(d). **Answer:** $\boxed{\vec{a} \cdot \vec{b} \leq \|\vec{a}\| \left\|\vec{b}\right\|}$. We want to prove

$$a_1b_1 + a_2b_2 + a_3b_3 \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$$

This is true if its square is true; that is, if

$$a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3 \leq \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

The right-hand side expands to

$$a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_3^2 + a_3^2b_1^2 + a_2^2b_3^2 + a_3^2b_2^2$$

The square terms $(a_ib_i)^2$ for $i = 1, 2, 3$ cancel, so we need to prove the inequality

$$2a_1b_1a_2b_2 + 2a_1b_1a_3b_3 + 2a_2b_2a_3b_3 \leq a_1^2b_2^2 + a_2^2b_1^2 + a_1^2b_3^2 + a_3^2b_1^2 + a_2^2b_3^2 + a_3^2b_2^2$$

According to the technique we used on the last worksheet, this is true since

$$2a_1b_1a_2b_2 \leq a_1^2b_2^2 + a_2^2b_1^2$$

and

$$2a_1b_1a_3b_3 \leq a_1^2b_3^2 + a_3^2b_1^2$$

and

$$2a_2b_2a_3b_3 \leq a_2^2b_3^2 + a_3^2b_2^2$$

all by difference of squares.[1] Therefore the dot product is less than the product of the magnitudes; that is, $\boxed{\vec{a} \cdot \vec{b} \leq \|\vec{a}\| \left\|\vec{b}\right\|}$. This is an expected result since

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$$

and

$$|\cos\theta| \leq 1$$

[1]There is a reason why this computation is so similar to one on the last worksheet. The computations were actually equivalent assuming the formula

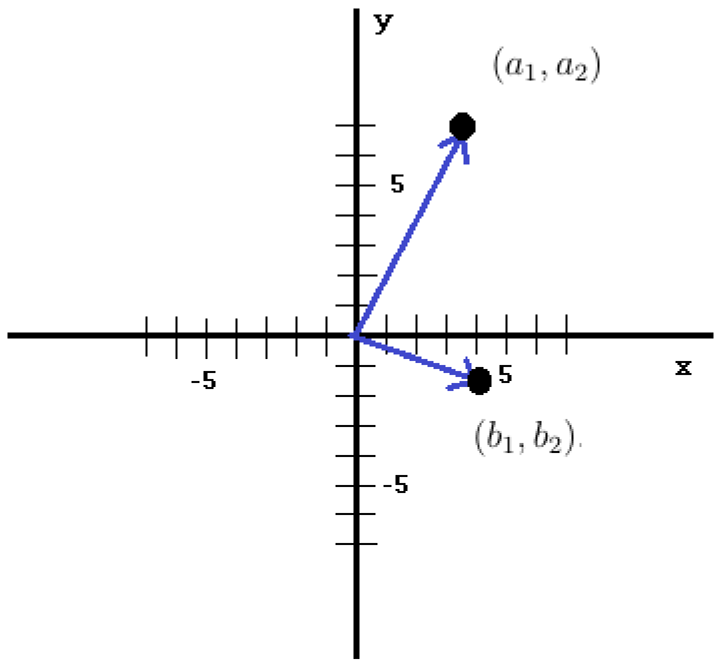
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

6. Working with the algebraic and trigonometric definitions of the dot product:

6(a). **Answer:** $\boxed{a_1b_1 + a_2b_2 = 0}$. If $\vec{a}$ and $\vec{b}$ are perpendicular, there is a right triangle formed with $\vec{a}$ and $\vec{b}$ as legs. The vertices of the triangle are $(0,0)$; (a_1, a_2); and (b_1, b_2). The lengths of the sides are

$$\sqrt{a_1^2 + a_2^2}, \sqrt{b_1^2 + b_2^2}, \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

where the last side, the hypotenuse, is the distance between (a_1, a_2) and (b_1, b_2).



Plot for 6a

Then, according to the Pythagorean theorem,

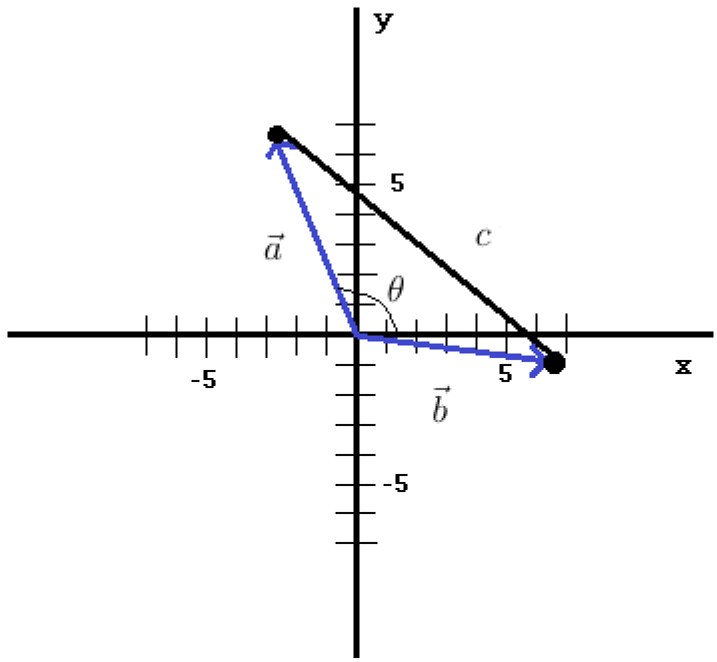
$$a_1^2 + a_2^2 + b_1^2 + b_2^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

Expanding the square terms on the right-hand side, we see that

$$2a_1b_1 + 2a_2b_2 = 0$$

which means that, if $\vec{a}$ and $\vec{b}$ are perpendicular, $\boxed{a_1b_1 + a_2b_2 = 0}$. This is the same perpendicularity formula as the condition $\vec{a} \cdot \vec{b} = 0$.

6(b). **Answer:** $\boxed{a_1b_1 + a_2b_2 = \|a\|\,\|b\| \cos\theta}$. To use the Law of Cosines, we need to draw a diagram of the triangle: the sides a and b corresponding to the vectors $\vec{a}$ and $\vec{b}$, and the side c which goes from (a_1, a_2) to (b_1, b_2).



Plot for 6b

The Law of Cosines indicates that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

The length of c is the distance from (a_1, a_2) to (b_1, b_2), which is

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

The value of a is the length of the vector $\vec{a}$, which is

$$\sqrt{a_1^2 + a_2^2}$$

Likewise, b is the length of $\vec{b}$, which is

$$\sqrt{b_1^2 + b_2^2}$$

Then, according to the Law of Cosines,

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}\cos\theta$$

Expanding the left-hand side, we see that

$$a_1^2 + b_1^2 - 2a_1b_1 + a_2^2 + b_2^2 - 2a_2b_2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}\cos\theta$$

Cancelling the square terms gives

$$-2a_1b_1 - 2a_2b_2 = -2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}\cos\theta$$

or

$$a_1b_1 + a_2b_2 = \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}\cos\theta$$

We identify

$$\sqrt{a_1^2 + a_2^2} = \|\vec{a}\|$$

and

$$\sqrt{b_1^2 + b_2^2} = \left\|\vec{b}\right\|$$

proving that $\boxed{a_1b_1 + a_2b_2 = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta}$ which is the geometric interpretation of the dot product.

6(c). **Answer:** $\boxed{a_1b_1 + a_2b_2 + a_3b_3 = \|a\| \, \|b\| \cos\theta}$. The proof is much the same as in the two-dimensional case. We again use the Law of Cosines. Of course, the triangle formed by $(0,0,0)$, (a_1,a_2,a_3), and (b_1,b_2,b_3) is not in the xy plane. However, according to the axioms of geometry, there is some plane that contains all three of those points. In that plane we can do geometry, much as we can in the xy plane. In that plane the Law of Cosines holds:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where θ is measured in that plane. Here, c is the distance between (a_1,a_2,a_3) and (b_1,b_2,b_3), which is

$$c = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$$

The number a is the length of $\vec{a}$, which is

$$\sqrt{a_1^2 + a_2^2 + a_3^2}$$

Likewise,

$$b = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Then we have that

$$(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2 = a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2-2\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}\cos\theta$$

Expanding the left-hand side, we see that all the square terms cancel, leaving us with

$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}\cos\theta$$

Again we have that

$$\sqrt{a_1^2+a_2^2+a_3^2} = \|\vec{a}\|$$

and

$$\sqrt{b_1^2+b_2^2+b_3^2} = \left\|\vec{b}\right\|$$

so we have that $\boxed{a_1b_1 + a_2b_2 + a_3b_3 = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta}$ which is the familiar trigonometric formula for dot product.